Math708 - Programming Project

1. (20 points) Let S be (a) the spline of degree 1; (b) the spline of degree 2; (c) the natural cubic spline function; (d) the polynomial of degree 40, which interpolates $f(x) = (1+x^2)^{-1}$ at 41 equally spaced knots in the interval$[-5, 5]$. Plot the functions $S(x)$ and $f(x)$, and evaluate $S(x) - f(x)$ at 101 equally spaced points on the interval $[0, 5]$.

2. (20 points) Compute an approximate value of the integral by (a) composite *Trapezoid* rule; (b) composite $Simpson's$ rule; (c) adaptive $Simpson's$ rule,

   $$\int_0^1 e^{-x^2} dx$$

   Here the uniform step size $h = 10^{-3}$.

3. (20 points) Find the root of the equation

   $$2x(1 - x^2 + x) \ln x = x^2 - 1$$

   in the interval $[0, 1]$ by (a) *bisection*; (b) *Newton's*; (c) *Secant* methods. Here the error tolerance is 10^{-5}. Show the running results in each step, plot $(x_n, f(x_n))$, and compare the efficiency (number of steps needed) for each method.

4. (20 points) Use (a): Newton's (b) Broyden's method to solve the following nonlinear system,

   $$f_1(x_1, x_2) = x_1^2 + x_2^2 - 2 = 0, \quad f_2(x_1, x_2) = x_1 - x_2 = 0.$$

   Verify that the system has two different solutions by changing initial guess, and verify the convergence rate for each method.

5. (20 points) Define an $n \times n$ matrix by $a_{i,j} = -1 + 2\max\{i, j\}$, and let $b_j = \sum_{j=1}^{n} a_{i,j}$. Here $n = 15$.
   (a): Test procedure $Naive - Gauss$ on this system;
   (b): Test procedure *Gaussian Elimination with Scaled Partial Pivoting* on this system;
   (c): Find LU factorization of matrix A, then use forward and backward elimination to solve $\mathbf{Ax} = \mathbf{b}$.
   (d): Test Conjugate Gradients (CG) method on the minimization of the quadratic function $F(x) = x^T Ax/2 - x^T b$.

6. (20 points) Solve the initial-value problem $x' = x/t + t\sec(x/t)$ with $x(0) = 0$ by (a) forward and (b) backward *Euler's* method; (c) the fourth-order *Runge − Kutta* method. Continue the solution to $t = 1$ using step sizes $h = 2^{-2}, 2^{-4}$ and 2^{-7}. Compare and plot the numerical solution with the exact solution, which is $x(t) = t \arcsin t$.