Riemannian Embedding behind Unnormalized Optimal Transport

In this section, for readers who are interested in physical interpretations, we present the Riemannian geometry calculus behind it. In a word, the proposed metric is induced by a Riemannian embedding, from Wasserstein-2 metric in normalized density space to the one in unnormalized density space.

Theorem 1 (Riemannian embedding)**.** *The unnormalized Wasserstein-2 metric tensor in $\mathcal{M}(\Omega)$ constrained to $\mathcal{P}(\Omega)$ is the Wasserstein-2 metric tensor.*

Proof. We first prove that our proposed new variational problem is a geometry energy (action) function in unnormalized density space. Consider a unnormalized density path

$$\partial_t\mu(t,x) + \nabla\cdot(\mu(t,x)v(t,x)) = f(t),$$

with $|\Omega| = 1$ and zero flux condition $\int_\Omega \mu v dx = 0$. Then $f(t) = \int_\Omega \partial_t\mu(t,x)dx$. In addition, from the Hodge decomposition on unnormalized density $\mu(t,x)$, then

$$v(t,x) = \nabla\Phi(t,x) + \Psi(t,x), \quad \text{where } \nabla\cdot(\mu(t,x)\Psi(t,x)) = 0.$$

Since $\mu \in \mathcal{M}_+(\Omega)$, we can identify the density path direction $\partial_t\mu$ with Φ by

$$\Phi(t,x) = (-\Delta_\mu)^{-1}\Big(\partial_t\mu(t,x) - \int_\Omega \partial_t\mu(t,x)dx\Big),$$

where $\Delta_\mu = \nabla\cdot(\mu\nabla)$ is the elliptic operator weighted by density μ. Thus

$$\begin{aligned}
&\int_0^1\int_\Omega \|v(t,x)\|^2\mu(t,x)dxdt + \frac{1}{\alpha}\int_0^1 |f(t)|^2 dt\\
=&\int_0^1\int_\Omega \|v(t,x)\|^2\mu(t,x)dxdt + \frac{1}{\alpha}\int_0^1 \Big(\int_\Omega \partial_t\mu(t,x)dx\Big)^2 dt\\
\geq&\int_0^1\int_\Omega \|\nabla\Phi(t,x)\|^2\mu(t,x)dxdt + \frac{1}{\alpha}\int_0^1 \Big(\int_\Omega \partial_t\mu(t,x)dx\Big)^2 dt\\
=&\int_0^1\int_\Omega \Big(\Phi(t,x), (-\Delta_{\mu_t})\Phi(t,x)\Big)dxdt + \frac{1}{\alpha}\int_0^1 \Big(\int_\Omega \partial_t\mu(t,x)dx\Big)^2 dt\\
=&\int_0^1\int_\Omega G_{\mathrm{UW}}(\mu)(\partial_t\mu_t, \partial_t\mu_t).
\end{aligned}$$

where

$$\begin{aligned}
G_{\mathrm{UW}}(\mu)(\partial_t\mu_t, \partial_t\mu_t) =& \int_0^1\int_\Omega \Big(\partial_t\mu_t - \int_\Omega \mu_t dx, (-\Delta_{\mu_t})^{-1}(\partial_t\mu_t - \int_\Omega \partial_t\mu_t dx)\Big)dxdt\\
&+ \frac{1}{\alpha}\int_0^1 \Big(\int_\Omega \partial_t\mu_t dx\Big)^2 dt.
\end{aligned} \tag{1}$$

Thus we have shown that

$$\mathrm{UW}_2(\mu_0,\mu_1)^2 = \inf_\mu \Big\{\int_0^1 G_{\mathrm{UW}}(\partial_t\mu_t, \partial_t\mu_t)dt \colon \mu(0,x) = \mu_0(x),\ \mu(1,x) = \mu_1(x)\Big\}.$$

The above is an action functional, with inner product G_{UW}.

We next prove that the inner product restricted in normalized density space is the classical L^2 Wasserstein inner product. If μ_t is in the normalized density space, i.e. consider $\int_\Omega \dot\mu dx = 0$, then

$$
\begin{aligned}
G_{\mathrm{UW}}(\mu)(\dot{\mu}, \dot{\mu}) =& \int_{\Omega} \Big(\dot{\mu}, (-\Delta_{\mu})\dot{\mu}\Big) dx \\
=& \Big\{ \int_{\Omega} \Big(\nabla\Phi(x), \nabla\Phi(x)\Big)\mu(x) dx \colon \dot{\mu} = -\Delta_{\mu}\Phi(x) \Big\}.
\end{aligned}
$$

This metric tensor coincides with the one in classical optimal transport [2]. This completes the proof. □

The unnormalized Wasserstein-2 metric tensor has been proposed in [1]. This is another motivation of the paper.

References

[1] W. Li. Geometry of probability simplex via optimal transport. *arXiv:1803.06360 [math]*, 2018.

[2] C. Villani. *Optimal Transport: Old and New*. Number 338 in Grundlehren der mathematischen Wissenschaften. Springer, Berlin, 2009.