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Research of combined hybrid method applied in the Reissner–Mindlin plate model

Zhu Wang *, Bing Hu

Department of Mathematics, Sichuan University, Chengdu 610064, PR China

Abstract

This paper has introduced two kinds of combined hybrid variational formulations for plate bending finite elements based upon the Reissner–Mindlin theory, according to whether assumed constant moment stress is introduced when assumed constant shear stress has been introduced. Due to two options for incompatible displacement modes, four new types of combined hybrid elements are proposed. A set of standard tests for both thin and thick plates show that new elements perform well.

Keywords: Reissner–Mindlin; Combined hybrid method; Assumed moment stress; Assumed shear stress; Incompatible displacement; Bubble

1. Introduction

Plate bending solutions pose a difficult problem for the classical finite element displacement method. Many element methods fail when the thickness of the plate is too small due to a phenomenon known as shear locking.

How to construct a general class of elements which can avoid shear locking has been the focus of much research during the past decades. Now there are many methods, such as the mixed interpolated method (see [1]), partial projection method (see [2]), non-conforming method (see [3]) are free of locking and provide unified approximation. However, how to apply and develop these methods in Reissner–Mindlin plate mode needs more investments.

In 1990, Weissman and Taylor constructed two plate bending elements, named CRB1 and CRB2, which avoided shear locking at the element level successfully, by using Hellinger–Reissner principle and introducing an explicit coupling between interpolations of shear and moment stress resultant fields. By comparing with four-node selective reduced integration in [5], ‘both CRB1 and CRB2 yield good results for both thin and thick plates’ (see [4]). However, for some problems, the two elements do not perform well in energy norm, which is the more natural basis for comparison.

* Corresponding author.
E-mail address: zhu_w@sohu.com (Z. Wang).

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Combined hybrid method, based on a weighted average of two systems of saddle point conditions corresponding to domain-decomposed Hellinger–Reissner variational principle and its dual, is a new hybrid method proposed in Ref. [6]. By adjusting the combined parameter, this method always can make the system's energy optimal. It has been successfully introduced into many engineering problems (see [7,8], etc.).

Recently, In [8], Hu applied energy compatibility and Wilson incompatible displacement modes in the approach of Reissner–Mindlin plate model, presented and analyzed a lower order quadrilateral element, which is locking-free and of high accuracy only using few meshes.

In this paper combined hybrid method is used to formulate plate bending elements based upon Reissner–Mindlin plate theory, constant shear and moment stress resultant fields are independently introduced at same time. Using same incompatible displace modes as CRB1 and CRB2 (see [4]), we posed four kinds of elements named PWu, PTL, CHWu, CHTL. By comparing Energy norms, center transverse displacement and center moment with exact solutions we can find, Numerical experiments show they perform well in a set of test problems selected from the literature. Especially, in square and circle plates, CHWu gets 96% of the center transverse displacement, 100.3% of the energy and 99.6% of the center moment on average, which yields better results than both CRB1 and CRB2 in most of the tests.

This paper is organized as follows: in Section 2, we give two types of different variational formulations. in Section 3, we construct four sorts of elements, and numerical results can be found in Section 4.

2. Combined hybrid method

In this section we discuss the combined variational problems for the Reissner–Mindlin plate model in both continuous and discrete spaces, the existence, uniqueness and convergence of their solution are provided.

2.1. Variational formulation in continuous spaces

The variational problem for the minimum problem of the Reissner–Mindlin plate model is: Find $(\beta,\omega)\in(H_0^1(\Omega))^2\times H_0^1(\Omega)$, such that

$$a(\beta,\eta)+\lambda t^{-2}(\nabla\omega-\beta,\nabla v-\eta)=(f,v),\quad \forall(\eta,v)\in(H_0^1(\Omega))^2\times H_0^1(\Omega). \tag{2.1}$$

Given (β,ω), shear stress is

$$\sigma=\lambda t^{-2}(\nabla\omega-\beta) \tag{2.2}$$

and moment stress is

$$M=D(\epsilon(\beta)), \tag{2.3}$$

where

$$a(\beta,\eta)=\frac{E}{12(1-\mu^2)}\int_\Omega\left\{\mu\,\mathrm{div}\,\beta\,\mathrm{div}\,\eta+\frac{1-\mu}{4}\sum_{i,j=1}^{2}\left(\frac{\partial\beta_i}{\partial x_j}+\frac{\partial\beta_j}{\partial x_i}\right)\left(\frac{\partial\eta_i}{\partial x_j}+\frac{\partial\eta_j}{\partial x_i}\right)\right\}\mathrm{d}\Omega,$$

D is plate bending stiffness, $\epsilon(\beta)=\left[\frac{\partial\beta_x}{\partial x},\frac{\partial\beta_y}{\partial y},\frac{\partial\beta_x}{\partial y},\frac{\partial\beta_y}{\partial x}\right]^{\mathrm{T}}$.

By using the combined hybrid method and introducing shear stress as an independent variate at same time, the equivalent problems of (2.1) as follows:

Find $(\sigma,\omega,\beta)\in\Gamma\times U\times H$, such that

$$a(\beta,\eta)+\lambda\alpha\sum_i(\nabla\omega-\beta,\nabla v-\eta)_{\Omega_i}+(1-\alpha t^2)\sum_i(\sigma,\nabla v-\eta)_{\Omega_i}-b_1(\sigma,v_I)=(g,v)\quad\forall(\eta,v)\in H\times U, \tag{2.4}$$

$$(1-\alpha t^2)\lambda^{-1}t^2(\sigma,\tau)-(1-\alpha t^2)\sum_i(\tau,\nabla\omega-\beta)_{\Omega_i}+b_1(\tau,\omega_I)=0\quad\forall\tau\in\Gamma, \tag{2.5}$$

where $b_1(\tau,v)=\sum_i\oint_{\partial\Omega_i}(\tau,\vec{n}), H=(H_0^1(\Omega))^2, \Gamma=\prod_i H(\mathrm{div},\Omega_i), U=U_c\oplus U_I, U_c=H_0^1(\Omega), U_I(\Omega_i)=span$ $(bubbles), \alpha\in(0,1)$.

Next, we will introduce moment stress as well as shear stress as independent varieties, by using the combined hybrid method, the equivalent problems of (2.1) as follows: Find $(\sigma,M,\omega,\beta)\in\Gamma\times Z\times U\times H$, such that

$$(1-\alpha_2)a(\beta,\eta)+\lambda\alpha_1\sum_i(\nabla\omega-\beta,\nabla v-\eta)_{\Omega_i}+(1-\alpha_1t^2)\sum_i(\sigma,\nabla v-\eta)_{\Omega_i}+\alpha_2(M,\epsilon(\eta)) - b_1(\sigma,v_I)=(g,v)\quad\forall(\eta,v)\in H\times U, \tag{2.6}$$

$$(1-\alpha_1t^2)\lambda^{-1}t^2(\sigma,\tau)-(1-\alpha_1t^2)\sum_i(\tau,\nabla\omega-\beta)_{\Omega_i}+\alpha_2d(M,m)-\alpha_2(m,\epsilon(\beta))+b_1(\tau,\omega_I)=0\quad\forall\tau\in\Gamma, \tag{2.7}$$

where b_1,H,Γ,U are defined same as above, $d(M,m)=\int_\Omega MD^{-1}m\,\mathrm{d}\Omega, h_t=\max\{t,h\}, h=\max\{h_i\}, T_h=\{\Omega_i\}$ denotes a finite element regular subdivision of Ω, $\alpha_1\in(0,t^{-2})$, $\alpha_2\in(0,1)$ are combined parameters.

Lemma 2.1 (Existence and Uniqueness). *The combined variational problems (2.4), (2.5) exist a unique solution $(\sigma,\omega,\beta)\in\Gamma\times U\times H$.*

Proof. It is obvious that the solution to the original differential formulation of Reissner–Mindlin plate model is the solution to (2.4) and (2.5).

Let $g=0$, $\eta=\beta$, $\tau=\sigma$, $v=\omega$, $v_I=\omega_I$ in (2.4), (2.5), we have

$$\alpha(\beta,\beta)+\alpha\lambda\sum_i\|\nabla\omega-\beta\|^2_{0,\Omega_i}+(1-\alpha t^2)\lambda^{-1}t^2(\sigma,\sigma)=0. \tag{2.8}$$

Then we have $\nabla\omega|_{\Omega_i}=0$ and $\beta=0,\sigma=0$ in Ω. From (2.5), it follows that: for $\forall\tau\in\Gamma$, $b_1(\tau,\omega)=0$, where $\omega_I=\omega-\omega_c$. As for $(\omega,\omega_c)\in U\times U_c$, we have $\omega\in H_0^1(\omega)$, then $(\omega-\omega_c)|_{\partial\Omega_i}=0$ and $\omega=0$ [7] in ω. Till now, the existence and uniqueness are proved. □

Lemma 2.2. *The combined variational problems (2.6) and (2.7) exist a unique solution $(\sigma,M,\omega,\beta)\in\Gamma\times Z\times U\times H$.*

2.2. Discretization and convergence

Let H^h, U^h, Γ^h, Z^h be the finite element spaces associated with T_h such that: $H^h\subset H$, $U^h\subset U$, $\Gamma^h\subset\Gamma$, $Z^h\subset Z$. (2.4) and (2.5) can be discretized as follows:

Find $(\sigma_h,\omega_h,\beta_h)\in\Gamma^h\times U^h\times H^h$ such that:

$$a(\beta_h,\eta)+\lambda\alpha\sum_i(\nabla\omega_h-\beta_h,\nabla v-\eta)_{\Omega_i}+(1-\alpha t^2)\times\sum_i(\sigma_h,\nabla v-\eta)_{\Omega_i}-b_1(\sigma_h,v_I)=(g,v)\quad\forall(\eta,v)\in H_h\times U_h, \tag{2.9}$$

$$(1-\alpha t^2)\lambda^{-1}t^2(\sigma_h,\tau)-(1-\alpha t^2)\sum_i(\tau,\nabla\omega_h-\beta_h)_{\Omega_i}+b_1(\tau,\omega_{h_I})=0\quad\forall\tau\in\Gamma^h. \tag{2.10}$$

Now we analyze the existence, uniqueness and convergence of the solution to above problems:

Lemma 2.3. *Assume (σ,ω,β) be the exact solution to the original differential formulation of Reissner–Mindlin plate model, where $\sigma=t^{-2}(\nabla\omega-\beta)$, then (2.9), (2.10) exist a unique solution $(\sigma_h,\omega_h,\beta_h)\in\Gamma^h\times U^h\times H^h$, and*

$$\|\sigma-\sigma_h\|_{0,\Omega}+\|\beta-\beta_h\|_{1,\Omega}+\left(\sum_i\|\nabla(\omega-\omega_h)\|^2_{0,\Omega_i}\right)^{1/2} \leq C\left\{\inf_{\tau\in\Gamma}\|\sigma-\tau\|_\Gamma+\inf_{\eta\in H^h}\|\beta-\eta\|_{1,\Omega}+\inf_{v\in U^h}\left[\left(\|\omega-v\|^2_U+\|v-v_c\|^2_{\frac12,\partial\Omega}\right)\right]^{1/2}\right\}, \tag{2.11}$$

where $C>0$ is a constant independent of h.

Proof. From (2.9) and (2.10), we know $\alpha(\eta,\eta)+\alpha\sum_i(\nabla v-\eta,\nabla v-\eta)_{\Omega_i}+(1-\alpha t^2)t^2(\tau,\tau)$ is regular in $\Gamma^h\times U^h\times H^h$, so we can get the existence and uniqueness of the finite element solution upon the Lax–Milgram theory. The error estimate can be found in [9]. □

(2.6) and (2.7) can be discretized as follows:
Find $(\sigma_h, M_h, \omega_h, \beta_h)\in\Gamma^h\times Z^h\times U^h\times H^h$, such that

$$(1-\alpha_2)a(\beta_h,\eta)+\lambda\alpha_1\sum_i(\nabla\omega_h-\beta_h,\nabla v-\eta)_{\Omega_i}+(1-\alpha_1t^2)\sum_i(\sigma_h,\nabla v-\eta)_{\Omega_i}+\alpha_2(M_h,\epsilon(\eta))$$
$$-b_1(\sigma_h,v_I)=(g,v)\quad\forall(\eta,v)\in H\times U,\tag{2.12}$$
$$(1-\alpha_1t^2)\lambda^{-1}t^2(\sigma_h,\tau)-(1-\alpha_1t^2)\sum_i(\tau,\nabla\omega_h-\beta_h)_{\Omega_i}+\alpha_2d(M_h,m)-\alpha_2(m,\epsilon(\beta_h))+b_1(\tau,\omega_{H_I})=0\quad\forall\tau^h\in\Gamma.\tag{2.13}$$

As to the existence, uniqueness and convergence of the solution to above problems, we have

Lemma 2.4. *Assume (σ, M,ω,β) be the exact solution to the original differential formulation of Reissner–Mindlin plate model, where $\sigma=t^{-2}(\nabla\omega-\beta)$, $M=D\epsilon(\beta)$, then (2.12), (2.13) exists a unique solution $(\sigma_h, M_h,\omega_h,\beta_h)\in\Gamma^h\times Z^h\times U^h\times H^h$, and*

$$\|\sigma-\sigma_h\|_{0,\Omega}+\|\beta-\beta_h\|_{1,\Omega}+\left(\sum_i\|\nabla(\omega-\omega_h)\|^2_{0,\Omega_i}\right)^{1/2}$$
$$\leqq C\left\{\inf_{\tau\in\Gamma}\|\sigma-\tau\|_\Gamma+\inf_{\eta\in H^h}\|\beta-\eta\|_{1,\Omega}+\inf_{v\in U^h}\left[\left(\|\omega-v\|^2_U+\|v-v_c\|^2_{\frac{1}{2},\partial\Omega}\right)\right]^{1/2}\right\},\tag{2.14}$$

where $C>0$ is a constant independent of h.

3. New elements

3.1. Assumed shear resultant field

In this paper we assume concretely that $\Gamma^h:=\{\tau\in\Gamma:\tau|_{\Omega_i}=\text{constant}\forall\Omega_i\in T_h\}$, i.e.

$$\tau|_{\Omega_i}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix},$$

where c_1, c_2 are constant.

3.2. Assumed moment modes

In this work we assume concretely that $Z^h:=\{m\in Z:m|_{\Omega_i}=\text{constant}\forall\Omega_i\in T_h\}$, i.e.

$$m|_{\Omega_i}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix},$$

where c_1, c_2, c_3 are constant.

3.3. Assumed incompatible modes

Compatible displacement $\hat{v}_c=\hat{v}(F_i(\xi,\eta))=[N_1\quad N_2\quad N_3\quad N_4]q_c^{(v)}$, where $q_c^{(v)}=(v_1\quad v_2\quad v_3\quad v_4)^T$ is nodal displacement vector, $N_t=(1+\xi_t\xi)(1+\eta_t\eta)/4$, and $(\xi_t,\eta_t)(t=1,2,3,4)$ are $(-1,-1)$,$(1,-1)$,$(1,1)$ and

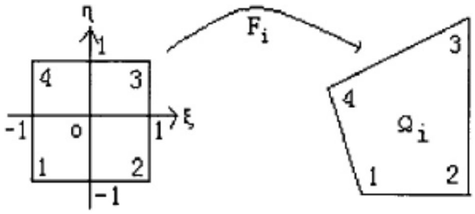

Fig. 1. Bilinear isoparameter mapping.

$(-1,1)$. $\begin{pmatrix} x \\ y \end{pmatrix} = F_i(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)\begin{pmatrix} x_i \\ y_i \end{pmatrix}$ is the isoparameteric mapping from the referential square $\hat{k}:[-1,1]\times[-1,1]$ to Ω_i (see Fig. 1) and $(x_t, y_t)(t = 1,2,3,4)$ are the co-ordinates of the four vertices.

Two options for the incompatible modes, were used in [4], are presented leading two elements.

The first type was presented by Taylor et al. [10], we use the function given by Weissman and Taylor in [4] as

$$\hat{v}_I{}^{TL} = [N_1^i \quad N_2^i] q_I^{(v)},$$

where

$$N_1^i = \left(1 - \frac{J_2}{J_0}\eta\right)(1-\xi^2) + \frac{J_1}{J_0}\eta(1-\xi^2),$$
$$N_2^i = \left(1 - \frac{J_1}{J_0}\xi\right)(1-\eta^2) + \frac{J_2}{J_0}\eta(1-\xi^2) \quad \forall q_I^v \in R^4.$$

The second type was presented by Pian and Wu et al. [11], we also use the function given by Weissman and Taylor in [4] as

$$\hat{v}_I^{Wu} = [N_3^i \quad N_4^i] q_I^{(v)},$$

where

$$N_3^i = \xi^2 - \frac{2J_1}{3J_0}\xi + \frac{2J_2}{3J_0}\eta,$$
$$N_4^i = \eta^2 + \frac{2J_1}{3J_0}\xi - \frac{2J_2}{3J_0}\eta \quad \forall q_I^v \in R^4.$$

where $J_0 = a_1b_3 - a_3b_1,\ J_1 = a_1b_2 - a_2b_1,\ J_2 = a_2b_3 - a_3\ b_2$. and $a_1 = \frac{1}{4}(-x_1 + x_2 + x_3 - x_4), a_2 = \frac{1}{4}(x_1 - x_2 + x_3 - x_4), a_3 = \frac{1}{4}(-x_1 - x_2 + x_3 + x_4)$. $\quad b_1 = \frac{1}{4}(-y_1 + y_2 + y_3 - y_4), b_2 = \frac{1}{4}(y_1 - y_2 + y_3 - y_4), b_3 = \frac{1}{4}(-y_1 - y_2 + y_3 + y_4)$.

3.4. New finite elements and convergence

Let

$$\Gamma^h := \{\tau \in \Gamma : \tau|_{\Omega_i} = \text{constant} \forall \Omega_i \in T_h\},$$
$$Z^h := \{m \in Z : m|_{\Omega_i} = \text{constant} \forall \Omega_i \in T_h\},$$
$$U_{TL}^h := \{v \in U : v|_{\Omega_i} = (v_c + v_I^{TL})|_{\Omega_i} = [\hat{v}_c + \hat{v}_I{}^{TL}] \circ F_i^{-1}, \forall \Omega_i \in T_h\},$$
$$U_{Wu}^h := \{v \in U : v|_{\Omega_i} = (v_c + v_I^{Wu})|_{\Omega_i} = [\hat{v}_c + \hat{v}_I^{Wu}] \circ F_i^{-1}, \forall \Omega_i \in T_h\},$$
$$H^h := \{\eta \in H : \eta|_{\Omega_i} \in (Q_1(\Omega_i))^2 \forall \Omega_i \in T_h\}.$$

Define four sets of new quadrilateral finite elements spaces as follows: The combined hybrid elements, corresponding to $\Gamma^h \times U^h_{TL} \times H^h$, $\Gamma^h \times U^h_{Wu} \times H^h$, $\Gamma^h \times Z^h \times U^h_{TL} \times H^h$, $\Gamma^h \times Z^h \times U^h_{Wu} \times H^h$ will be denoted by PTL, PWu, CHTL, CHWu.

Lemma 3.1. *When the mesh of discretization is convex quadrangle, for every* $\tau \in \Gamma^h$, $v \in U^h_{TL}$ *or* $v \in U^h_{Wu}$, $b_1(\tau, v) = 0$, *is completely energy compatible.*

Proof

$$b_1(\tau, v_I) = \sum_i \oint_{\partial\Omega_i} \tau \cdot \vec{n} \cdot v_I \,\mathrm{d}s = \sum_i (\operatorname{div}\tau, v_I)_{\Omega_i} + (\tau, \nabla v_I)_{\Omega_i} = \tau \sum_k \int_{\hat{k}} [J]^{-1} \begin{pmatrix} \frac{\partial v_I}{\partial \xi} \\ \frac{\partial v_I}{\partial \eta} \end{pmatrix} \mathrm{d}k = 0. \qquad \square$$

Theorem 3.1. *Assume* $\omega \in H^1_0(\Omega) \cap H^3(\Omega)$ *and* $\beta \in (H^1_0(\Omega) \cap H^2(\Omega)^2)$, *then the unique solution*, $(\sigma_h, \omega_h, \beta_h)$ *determined by PTL, to the problems (2.9) and (2.10), satisfies*

$$t\|\sigma - \sigma_h\|_{0,\Omega} + \|\beta - \beta_h\|_{1,\Omega} + \left(\sum_i \|\nabla(\omega - \omega_h) - (\beta - \beta_h)\|^2_{0.\Omega_i}\right)^{\frac{1}{2}}$$
$$\leqslant C\{h\|\beta\|_{2,\Omega} + (h^2 + h^2/t)(\|\omega\|_{3,\Omega} + \|\beta\|_{2,\Omega}) + h(1+t)\|\sigma\|_{1,\Omega} + h\|\operatorname{div}\sigma\|_{0,\Omega}\}, \tag{3.1}$$

where C is independent of h, α, *the poisson ratio* v.

Proof. First, the existence and uniqueness of the solution is obvious (Lemma 2.1).

Second, assume $(\Pi_0\beta, \Pi_1\omega, \Pi_2\sigma) \in U^h_{TL} \times H^h \times \Gamma^h$, is any given approximation of (β, ω, σ).

Setting up $\eta = \delta\beta_h := \Pi_0\beta - \beta_h$, $v = \delta\ \omega_h := \Pi_1\omega - \omega_h$, $\tau = \delta\sigma_h := \Pi_2\sigma - \sigma_h$, and subtracting Eqs. (2.9), (2.10) from (2.6), (2.7), we have

$$\Sigma := a(\delta\beta_h, \delta\beta_h) + \alpha\lambda \sum_i \|\nabla(\delta\omega_h) - \delta\beta_h\|^2_{0,\Omega_i} + (1 - \alpha t^2)\lambda^{-1} t^2 \|\delta\sigma_h\|^2_{0,\Omega} = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 := a(\Pi_0\beta - \beta, \delta\beta_h) + \alpha\lambda \sum_i (\nabla(\Pi_1\omega - \omega) - (\Pi_0\beta - \beta), \nabla(\delta\omega_h) - \delta\beta_h)_{\Omega_i} + (1 - \alpha t^2)\lambda^{-1} t^2 (\Pi_2\sigma - \sigma, \delta\sigma_h),$$
$$I_2 := (1 - \alpha t^2) \sum_i (\Pi_2\sigma - \sigma, \nabla(\delta\omega_h) - \delta\beta_h)_{\Omega_i},$$
$$I_3 := -(1 - \alpha t^2) \sum_i (\delta\sigma_h, \nabla(\Pi_1\omega - \omega) - (\Pi_0\beta - \beta))_{\Omega_i},$$
$$I_4 := b_1(\delta\sigma_h, \Pi_1\omega - (\Pi_1\omega)_c),$$
$$I_5 := -b_1(\Pi_2\sigma - \sigma, \delta\omega_h - (\delta\omega_h)_c).$$

By using Schwarz' Inequality, we have

$$I_1 \leqslant C\left[\|\Pi_0\beta - \beta\|_1 + \left(\sum_i \|\nabla(\Pi_1\omega - \omega) - (\Pi_0\beta - \beta)\|^2_{0,\Omega_i}\right)^{1/2} + t\|\Pi_2\sigma - \sigma\|_{0,\Omega}\right] \times (\Sigma)^{1/2},$$
$$I_2 \leqslant C\|\Pi_2\sigma - \sigma\|_{0,\Omega} \times \left(\alpha \sum_i \|\nabla(\delta\omega_h) - \delta\beta_h\|^2_{0,\Omega_i}\right)^{1/2},$$
$$I_3 \leqslant C \times (1/t)\left(\sum_i \|\nabla(\Pi_1\omega - \omega) - (\Pi_0\beta - \beta)\|^2_{0,\Omega_i}\right)^{1/2} \times ((1 - \alpha t^2) t^2 \|\delta\sigma_h\|^2_{0,\Omega})^{1/2},$$
$$I_4 = 0 \text{ (Lemma 3.1)},$$
$$I_5 \leqslant C\|\Pi_2\sigma - \sigma\|_{\Gamma} \times (\Sigma)^{1/2}.$$

From all the above inequations,

$$\left(a(\delta\beta_h,\delta\beta_h)+\alpha\sum_i\|\nabla(\delta\omega_h)-\delta\beta_h\|^2_{0,\Omega_i}+(1-\alpha t^2)t^2\|\delta\sigma_h\|^2_{0,\Omega}\right)^{1/2}$$
$$\leqslant C\left\{\|\Pi_0\beta-\beta\|_1+(1+1/t)\left(\sum_i\|\nabla(\Pi_1\omega-\omega)-(\Pi_0\beta-\beta)\|^2_{0,\Omega_i}\right)^{1/2}+(1+t)\|\Pi_2\sigma-\sigma\|_{0,\Omega}+\|\Pi_2\sigma-\sigma\|_\Gamma\right\}.$$

By using triangular inequality and interpolation error estimate, we deduce that

$$\|\beta-\beta_h\|_1+\left(\sum_i\|\nabla(\omega-\omega_h)-(\beta-\beta_h)\|^2_{0,\Omega_i}\right)^{1/2}+t\|\sigma-\sigma_h\|_{0,\Omega}$$
$$\leqslant C(h\|\beta\|_{2,\Omega}+(h^2+h^2/t)(\|\omega\|_{3,\Omega}+\|\beta\|_{2,\Omega})+h(1+t)\|\sigma\|_{1,\Omega}+h\|div\sigma\|_{0,\Omega}. \qquad \square$$

Remark. It is easily proved that the second element (PWu) has the same error estimate by same technique.

Theorem 3.2. *Assume* $\omega\in H^1_0(\Omega)\cap H^3(\Omega)$ *and* $\beta\in(H^1_0(\Omega)\bigcap H^2(\Omega))^2$, *then the unique solution*, $(\sigma_h,M_h,\omega_h,\beta_h)$ *determined by CHTL or CHWu, to the problems (2.11) and (2.12), satisfies*:

$$t\|\sigma-\sigma_h\|_{0,\Omega}+\|\beta-\beta_h\|_{1,\Omega}+\|M-M_h\|_{0,\Omega}+\left(\sum_i\|\nabla(\omega-\omega_h)-(\beta-\beta_h)\|^2_{0,\Omega_i}\right)^{\frac{1}{2}}$$
$$\leqslant C\{h\|\beta\|_{2,\Omega}+(h^2+h^2/t)(\|\omega\|_{3,\Omega}+\|\beta\|_{2,\Omega})+h(1+t)\|\sigma\|_{1,\Omega}+h\|\mathrm{div}\,\sigma\|_{0,\Omega}+h\|M\|_{1,\Omega}\}. \tag{3.2}$$

where C is independent of h,α, *the poisson ratio* v.

4. Numerical experiment

In this section some numerical results are presented. By using a set of problems selected from the literature, the performance of new finite elements: PTL, PWU, CHTL and CHWu are evaluated. The purpose of these evaluations is to test the combined hybrid variational formulations' sensitivity to the specific choice of incompatible modes as well as their overall performance. Evaluations are done with square plates, circular plates, a highly skewed rhombic plate and sensitivity to mesh distortion. All these results are compared with those given by Weissman and Taylor in [4].

Convergence in the energy norm is the natural convergence test for the finite element method (see [12]). It's common practice in the literature, however, to examine convergence of the finite element solution by analyzing the displacement at characteristic points. In this paper, convergence is examined in terms of energy norm, center displacement and moment. All tables show the center displacement, moment and energy norm as a function of number of elements (denoted nel) used in the corresponding mesh.

Only the case of uniform loading is examined.

4.1. Square plates

A square plate is modeled using meshes of uniform square elements. Due to symmetry, only one quadrant is discretized, and 4×4 mesh is shown in Fig. 2. Object to let the plate bending stiffness $D=1.0$, using the material properties in Table 1.

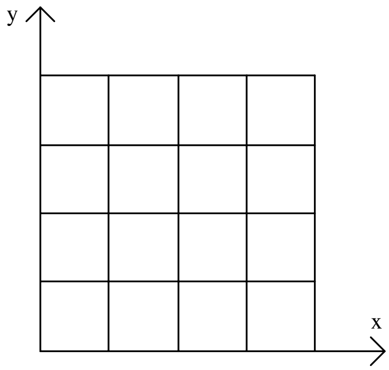

Fig. 2. 1/4 square plate (16 elements).

Table 1
Material properties

	Thin plate	Thick plate
E	10.92e+6	1.365
v	0.3	0.3
T	0.01	2
L	10.0	10.0

4.1.1. Thin plate

4.1.1.1. Clamped plate. Results are summarized in Table 2, and shown in Fig. 3. The exact solutions are $\omega_c = 12.6532$, $M_c = -2.2905$ (see [13]).

Four new elements perform better than both CRB1 and CRB2 in energy norm, because this method coerces energy to be optimal in order to get well results by adjusting α. All six elements yield nearly identical results. CHWu yields the best result for this problem, with 16 elements in the mesh, 99.95% of the center transverse displacement, and 102.80% of the moment is obtained.

4.1.1.2. Simply support plate. Results are summarized in Table 3, and shown in Fig. 4. The exact solutions are Energy = 425.62, $\omega_c = 40.6237$, $M_c = -4.78863$ (see [14]).

Table 2
Square plate—uniform load, clamped, $t = 0.01$

Mesh size		2 × 2	4 × 4	8 × 8	16 × 16	Solution
Energy	WT (CRB1)	75.75949	91.88611	96.59564	97.63482	
	WT (CRB2)	75.75084	91.84711	96.57340	98.36972	
	PWu ($\alpha = 0.080$)	98.3908	97.4356	97.3146	97.2905	
	PTL ($\alpha = 0.31$)	97.5591	97.5061	97.3539	97.3017	
	CHWu ($\alpha_1 = 0.10, \alpha_2 = 0.70$)	97.4130	97.3341	97.2962	97.2864	
	CHTL ($\alpha_1 = 0.36, \alpha_2 = 0.70$)	97.5203	97.7127	97.4168	97.3181	
ω_c	WT (CRB1)	12.11830	12.52712	12.68109	12.70854	12.6532
	WT (CRB2)	12.11691	12.52163	12.67157	12.75608	
	PWu ($\alpha = 0.080$)	12.0221	12.5015	12.6164	12.6442	
	PTL ($\alpha = 0.31$)	11.7690	12.4854	12.6154	12.6442	
	CHWu ($\alpha_1 = 0.10, \alpha_2 = 0.70$)	12.6097	12.6469	12.6526	12.6533	
	CHTL ($\alpha_1 = 0.36, \alpha_2 = 0.70$)	12.2961	12.6284	12.6514	12.6532	
$-M_c$	PWu ($\alpha = 0.080$)	2.5005	2.3296	2.3004	2.2929	2.2905
	PTL ($\alpha = 0.31$)	2.4478	2.3268	2.3002	2.2929	
	CHWu ($\alpha_1 = 0.10, \alpha_2 = 0.70$)	2.6227	2.3547	2.3070	2.2946	
	CHTL ($\alpha_1 = 0.36, \alpha_2 = 0.70$)	2.5575	2.3516	2.3068	2.2946	

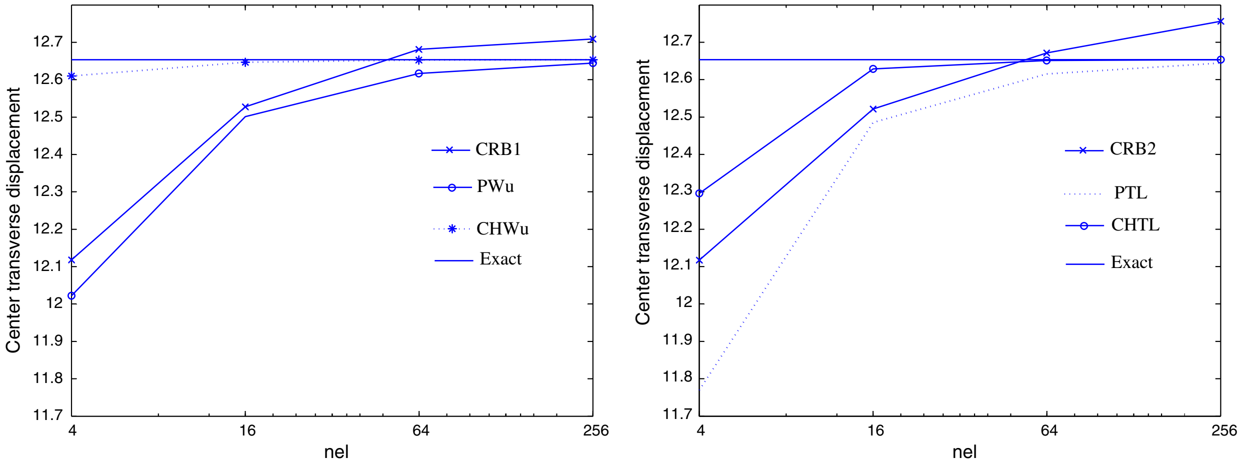

Fig. 3. Convergence study for a thin plate; clamped, uniform load.

Table 3
Square plate—uniform load, simply supported, $t = 0.01$

Mesh size		2×2	4×4	8×8	16×16	Solution
Energy	WT (CRB1)	398.80376	418.93874	424.12572	425.63930	425.62
	WT (CRB2)	389.90478	418.30261	424.07838	425.69397	
	PWu ($\alpha = 0.025$)	427.2153	426.6168	425.9665	425.8005	
	PTL ($\alpha = 1.9$)	426.4025	434.4322	429.0837	426.7542	
	CHWu ($\alpha_1 = 1.028, \alpha_2 = 0.60$)	427.0701	426.8238	426.0255	425.8154	
	CHTL ($\alpha_1 = 1.9, \alpha_2 = 0.010$)	426.5384	434.4697	429.0931	426.7565	
ω_c	WT (CRB1)	42.82542	41.14819	40.77255	40.70100	40.6237
	WT (CRB2)	42.24268	40.95346	40.71770	40.68911	
	PWu ($\alpha = 0.025$)	40.5090	40.7479	40.6661	40.6418	
	PTL ($\alpha = 1.9$)	40.6414	41.4308	40.9357	40.72402	
	CHWu ($\alpha_1 = 0.028, \alpha_2 = 0.60$)	41.3228	40.9603	40.7187	40.6548	
	CHTL ($\alpha_1 = 1.9, \alpha_2 = 0.010$)	40.6545	41.4343	40.9365	40.7242	
$-M_c$	PWu ($\alpha = 0.025$)	5.6619	5.0491	4.8564	4.8054	4.78863
	PTL ($\alpha = 1.9$)	4.8767	4.9461	4.8577	4.8107	
	CHWu ($\alpha_1 = 0.028, \alpha_2 = 0.60$)	5.8541	5.0714	4.8620	4.8069	
	CHTL ($\alpha_1 = 1.9, \alpha_2 = 0.010$)	4.8782	4.9465	4.8578	4.8107	

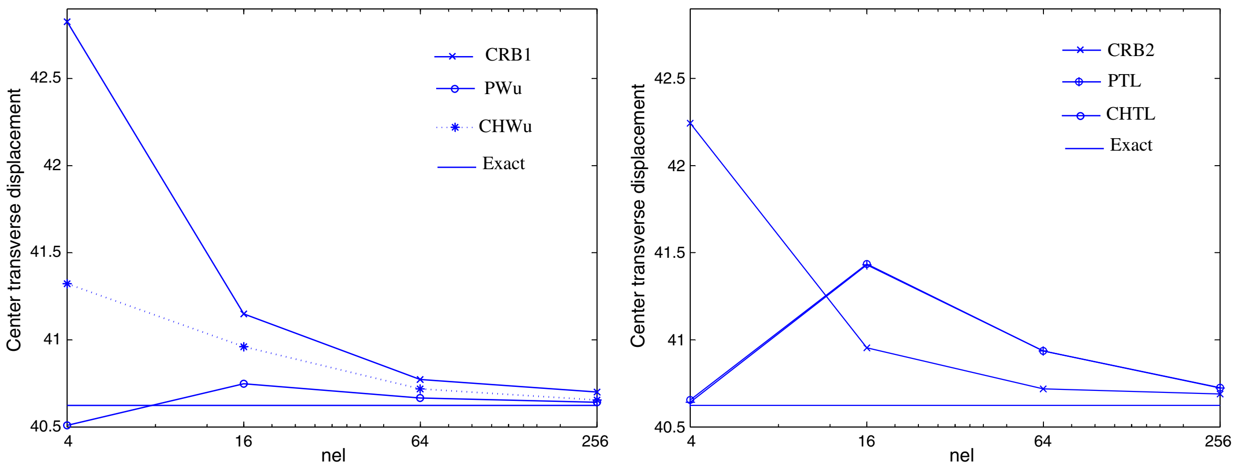

Fig. 4. Convergence study for a thin plate; simply supported, uniform load.

New elements perform little better than CRB1 and CRB2 in center transverse displacement. Especially, CHWu does converge monotonously in energy norm, center transverse displacement and moment, which yields the best result among six elements for this problem, with 16 elements in the mesh, 100.28% of the energy, 100.80% of the center transverse displacement, and 105.90% of the moment are obtained.

4.1.2. Thick plate

4.1.2.1. Clamped plate. Results are summarized in Table 4, and shown in Fig. 5.

Four new elements yield pretty better than both CRB1 and CRB2 in either energy norm or center transverse displacement. CHWu is obviously excellent for this problem. When 16-element mesh is used, CHWu gets 99.93% of energy, 100.19% of center transverse displacement and 100.38% of moment obtained for 256 elements.

4.1.2.2. Simply support plate. Results are summarized in Table 5, and shown in Fig. 6.

Four new elements yield a little better than both CRB1 and CRB2 in either energy norm or center transverse displacement. CHTL yields best result in center transverse displacement, while CHWu performs best in

Table 4
Square plate—uniform load, clamped, $t = 2$

Mesh size		2 × 2	4 × 4	8 × 8	16 × 16	Solution
Energy	WT (CRB1)	215.82077	208.66491	205.08661	204.09345	
	WT (CRB2)	300.06833	234.74044	211.82512	205.79011	
	PWu ($\alpha = 0.080$)	205.4944	204.1299	203.8409	203.7751	
	PTL ($\alpha = 0.22$)	203.6396	203.8019	203.7665	203.7569	
	CHWu ($\alpha_1 = 0.070, \alpha_2 = 0.45$)	203.0515	203.6072	203.7169	203.7447	
	CHTL ($\alpha_1 = 0.249, \alpha_2 = 0.99$)	203.8569	204.0631	203.8464	203.7781	
ω_c	WT (CRB1)	25.45333	22.84967	22.01024	21.79410	
	WT (CRB2)	32.89409	24.84375	22.51119	21.91939	
	PWu ($\alpha = 0.080$)	21.4517	21.6889	21.7145	21.7199	
	PTL ($\alpha = 0.22$)	20.8036	21.5781	21.6893	21.7137	
	CHWu ($\alpha_1 = 0.070, \alpha_2 = 0.45$)	21.7393	21.7668	21.7341	21.7248	
	CHTL ($\alpha_1 = 0.249, \alpha_2 = 0.99$)	21.4410	21.7446	21.7313	21.7243	
$-M_c$	PWu ($\alpha = 0.080$)	2.1480	2.3526	2.3568	2.3574	
	PTL ($\alpha = 0.22$)	2.1584	2.3484	2.3562	2.3572	
	CHWu ($\alpha_1 = 0.070, \alpha_2 = 0.45$)	2.1963	2.3672	2.3605	2.3583	
	CHTL ($\alpha_1 = 0.249, \alpha_2 = 0.99$)	2.2858	2.3761	2.3636	2.3592	

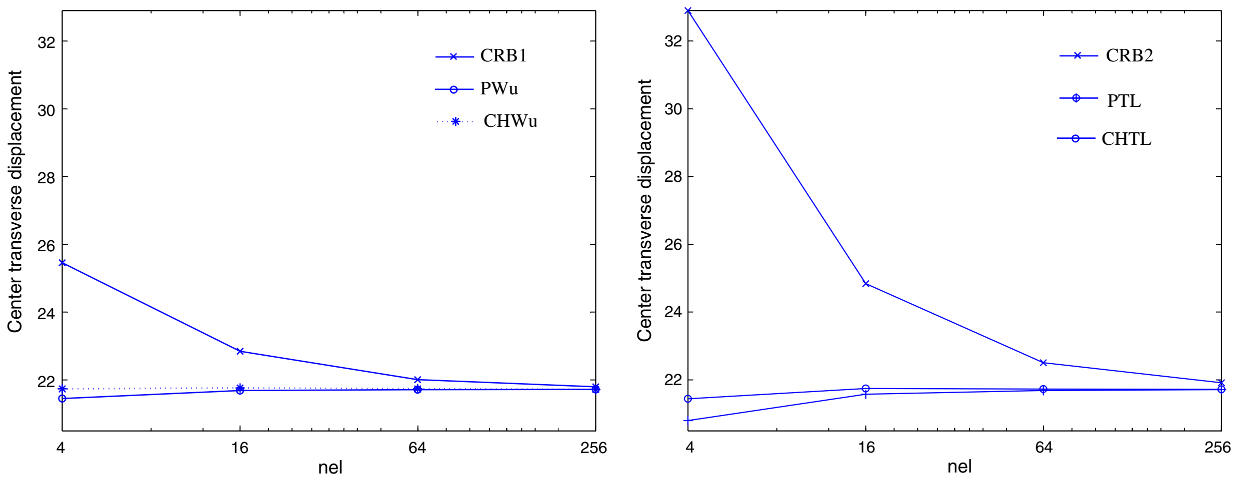

Fig. 5. Convergence study for a thick plate; clamped, uniform load.

Table 5
Square plate—uniform load, simply supported, $t = 2$

Mesh size		2 × 2	4 × 4	8 × 8	16 × 16	Solution
Energy	WT (CRB1)	551.24726	585.20271	596.38017	599.72511	
	WT (CRB2)	595.31737	610.13517	604.04344	601.75198	
	PWu ($\alpha = 0.015$)	606.4472	600.3107	600.6086	600.8267	
	PTL ($\alpha = 0.249$)	624.7536	602.1696	600.5462	600.7631	
	CHWu ($\alpha_1 = 0.016, \alpha_2 = 0.30$)	600.9708	599.1498	600.3497	600.7661	
	CHTL ($\alpha_1 = 0.249, \alpha_2 = 0.010$)	624.8595	602.1967	600.5540	600.7653	
ω_c	WT (CRB1)	58.61877	55.96186	55.52584	55.46432	
	WT (CRB2)	61.82475	57.76508	56.07810	55.61035	
	PWu ($\alpha = 0.015$)	53.5256	54.9481	55.3205	55.4174	
	PTL ($\alpha = 0.249$)	56.8934	55.4203	55.3865	55.4293	
	CHWu ($\alpha_1 = 0.016, \alpha_2 = 0.30$)	53.7819	55.0260	55.3420	55.4230	
	CHTL ($\alpha_1 = 0.249, \alpha_2 = 0.010$)	56.9042	55.4229	55.3873	55.4295	
$-M_c$	PWu ($\alpha = 0.015$)	5.6464	5.3166	5.3509	5.3596	
	PTL ($\alpha = 0.249$)	5.6582	5.3799	5.3617	5.3619	
	CHWu ($\alpha_1 = 0.016, \alpha_2 = 0.30$)	5.7192	5.3250	5.3533	5.3602	
	CHTL ($\alpha_1 = 0.249, \alpha_2 = 0.010$)	5.6598	5.3802	5.3618	5.3619	

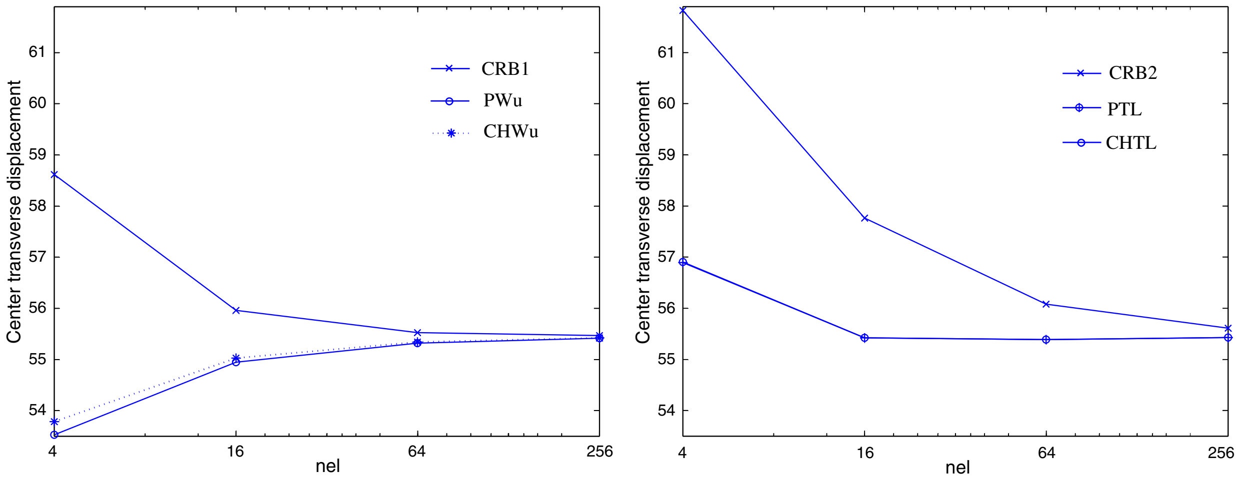

Fig. 6. Convergence study for a thick plate; simply supported, uniform load.

energy norm. What's more, CHWu gets 99.28% of center transverse displacement and 99.34% of moment obtained for 256 elements, when 16-element mesh is used.

4.2. Circular plates

A circular plate is modeled using 3, 12, 48, 192 elements. Due to symmetry, only one quadrant is calculated, and a discretization of 12 elements is shown in Fig. 7. Object to let the plate bending stiffness $D = 1.0$, using the material properties in Table 6.

4.2.1. Thin plate

4.2.1.1. Clamped plate. Results are summarized in Table 7, and shown in Fig. 8. The thin plate exact solutions are Energy = 64.09118, $\omega_c = 9.78348$ (see [4]), $M_c = -2.0313$ (see [13]).

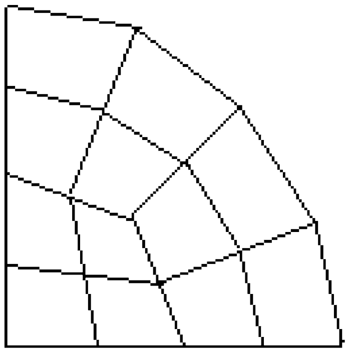

Fig. 7. Circular plate (12 elements).

Table 6
Material properties

	Thin plate	Thick plate
E	10.92e+6	1.365
v	0.3	0.3
T	0.01	2
L	10.0	10.0

Table 7
Circular plate—uniform load, clamped, $t = 0.1$

Mesh (nel)		3	12	48	192	Solution
Energy	WT (CRB1)	70.40178	68.44072	65.43534	64.43534	64.09118
	WT (CRB2)	113.87654	87.22176	70.86546	65.85591	
	PWu ($\alpha = 0.040$)	63.8559	64.8556	64.3306	64.1593	
	PTL ($\alpha = 0.15$)	64.2762	64.6226	64.2749	64.1447	
	CHWu ($\alpha_1 = 0.042, \alpha_2 = 0.99$)	64.2428	65.1478	64.4161	64.1802	
	CHTL ($\alpha_1 = 0.16, \alpha_2 = 0.90$)	63.9545	64.4760	64.1074	63.9701	
ω_c	WT (CRB1)	11.70388	10.24633	9.92960	9.82260	9.78348
	WT (CRB2)	15.36831	11.60705	10.29889	9.91879	
	PWu ($\alpha = 0.040$)	6.7667	8.9619	9.5841	9.7343	
	PTL ($\alpha = 0.15$)	6.8595	8.8966	9.5630	9.7287	
	CHWu ($\alpha_1 = 0.042, \alpha_2 = 0.99$)	7.2016	9.0864	9.6199	9.7433	
	CHTL ($\alpha_1 = 0.16, \alpha_2 = 0.90$)	7.3058	9.0014	9.5777	9.7190	
$-M_c$	PWu ($\alpha = 0.040$)	1.4906	1.9550	2.0175	2.0250	2.0313^+
	PTL ($\alpha = 0.15$)	1.5586	1.9610	2.0196	2.0258	
	CHWu ($\alpha_1 = 0.042, \alpha_2 = 0.99$)	1.6031	1.9687	2.0383	2.0277	
	CHTL ($\alpha_1 = 0.16, \alpha_2 = 0.90$)	1.7076	2.0096	2.0854	2.0638	

CHWu and CHTL yield good results for this problem. Especially, CHWu performs excellent among these elements: 92.5% of the exact displacement, 102.02% of the analyze energy, and 97.42% of the moment is obtained for 12 elements.

4.2.1.2. Simply support plate. Results are summarized in Table 8,and shown in Fig. 9. The exact solutions are Energy = 359.08748, $\omega_c = 39.83156$ (see [4]), $M_c = -5.1563$ (see [13]).

CRB2 gets the best results for this problem, which obtains 98.87% of the exact center transverse displacement and 96.34% of the exact energy for 12 elements. Among these new elements, CHTL performs best, which yields 96.69% of analyze center transverse displacement, 101.91% of analyze energy for 12 elements, while

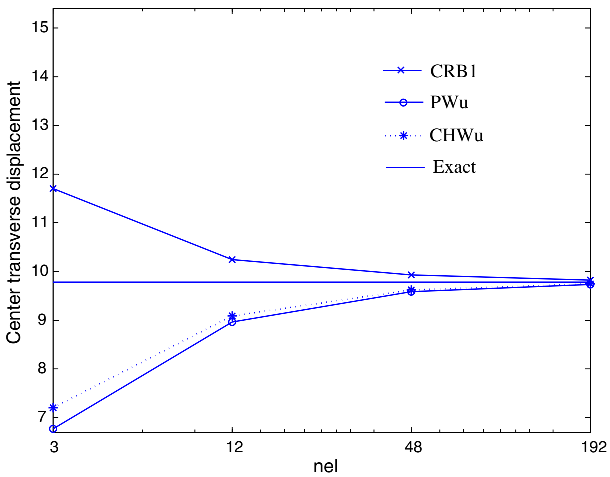

Fig. 8. Convergence study for a thin circular plate; clamped, uniform load.

Table 8
Circular plate—uniform load, simply supported, $t = 0.1$

Mesh size		2×2	4×4	8×8	16×16	Solution
Energy	WT (CRB1)	244.54439	327.44936	351.18592	357.11690	359.08748
	WT (CRB2)	282.52226	345.96120	356.66770	358.55009	
	PWu ($\alpha = 0.0070$)	355.2504	358.2662	358.8779	359.0403	
	PTL ($\alpha = 0.060$)	360.9260	366.0323	361.3407	359.6894	
	CHWu ($\alpha_1 = 0.0070, \alpha_2 = 0.99$)	357.2215	359.0631	359.0907	359.0929	
	CHTL ($\alpha_1 = 0.062, \alpha_2 = 0.99$)	360.2987	365.9547	361.3282	359.6850	
ω_c	WT (CRB1)	33.42998	38.25803	39.49348	39.74969	39.83156
	WT (CRB2)	36.50937	39.37985	39.84320	39.64496	
	PWu ($\alpha = 0.0070$)	27.5269	36.3716	38.9617	39.6138	
	PTL ($\alpha = 0.060$)	34.4101	38.3800	39.4894	39.7474	
	CHWu ($\alpha_1 = 0.070, \alpha_2 = 0.99$)	28.0146	36.4944	38.9971	39.6228	
	CHTL ($\alpha_1 = 0.062, \alpha_2 = 0.99$)	34.9410	38.5131	39.5250	39.7564	
$-M_c$	PWu ($\alpha = 0.0070$)	3.7760	4.8901	5.1122	5.1571	5.1563
	PTL ($\alpha = 0.060$)	4.5745	5.1227	5.1724	5.1517	
	CHWu ($\alpha_1 = 0.0070, \alpha_2 = 0.99$)	3.9762	4.8984	5.1368	5.1418	
	CHTL ($\alpha_1 = 0.062, \alpha_2 = 0.99$)	4.8178	5.1645	5.1990	5.1364	

CHWu performs excellent in energy norm, which yields 91.62% of center transverse displacement, 99.99% of the exact energy for 12 elements.

4.2.2. Thick plate

4.2.2.1. Clamped plate. Results are summarized in Table 9, and shown in Fig. 10. The exact solutions are Energy = 134.04070, $\omega_c = 16.90848$ (see [4]).

CRB1 gets excellent result for this problem among six elements: 102.68% of center transverse displacement, 99.65% of energy is obtained. CHTL gets the best result: 95.53% of center transverse displacement and 101.12% of the exact energy for 12 elements, while CHWu, with 12 elements in mesh, obtains 94.63% of the center transverse displacement and 101.37% of the exact energy.

4.2.2.2. Simply support plate. Results are summarized in Table 10, and shown in Fig. 11. The exact solutions are Energy = 429.03071, $\omega_c = 46.95656$ (see [4]).

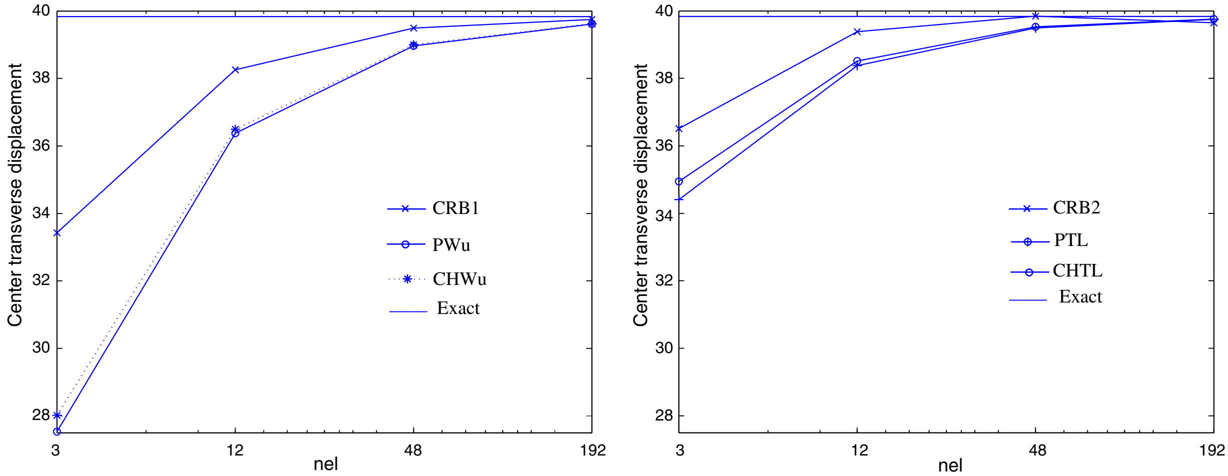

Fig. 9. Convergence study for a thin circular plate; simply supported, uniform load.

Table 9
Circular plate—uniform load, clamped, $t = 2$

Mesh (nel)		3	12	48	192	Solution
Energy	WT (CRB1)	123.25616	133.54089	134.10303	134.06845	134.04070
	WT (CRB2)	165.41479	152.52521	139.54992	135.47738	
	PWu ($\alpha = 0.025$)	133.9853	135.5052	134.5258	134.1700	
	PTL ($\alpha = 0.095$)	134.7764	135.6865	134.5691	134.1805	
	CHWu ($\alpha_1 = 0.025, \alpha_2 = 0.50$)	134.9163	135.8825	134.6293	134.1964	
	CHTL ($\alpha_1 = 0.10, \alpha_2 = 0.90$)	134.0601	135.5374	134.5353	134.1723	
ω_c	WT (CRB1)	18.29822	17.36156	17.02945	16.93933	16.90848
	WT (CRB2)	21.87458	18.73937	17.40752	17.03586	
	PWu ($\alpha = 0.025$)	13.0771	15.9319	16.6764	16.8510	
	PTL ($\alpha = 0.095$)	13.9146	16.0299	16.6786	16.8505	
	CHWu ($\alpha_1 = 0.025, \alpha_2 = 0.50$)	13.2875	16.0005	16.6944	16.8556	
	CHTL ($\alpha_1 = 0.10, \alpha_2 = 0.90$)	14.3638	16.1528	16.7109	16.8586	
$-M_c$	PWu ($\alpha = 0.025$)	1.5010	1.9213	2.0051	2.0247	
	PTL ($\alpha = 0.095$)	1.5553	1.9341	2.0079	2.0256	
	CHWu ($\alpha_1 = 0.025, \alpha_2 = 0.50$)	1.5559	1.9433	2.0111	2.0263	
	CHTL ($\alpha_1 = 0.10, \alpha_2 = 0.90$)	1.7304	1.9695	2.0190	2.0284	

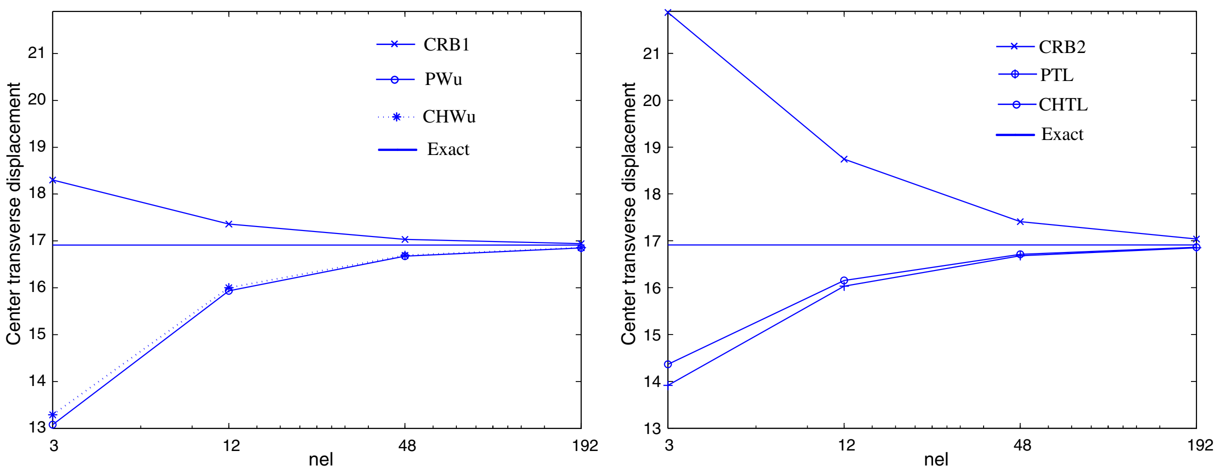

Fig. 10. Convergence study for a thick circular plate; clamped, uniform load.

Table 10
Circular plate—uniform load, simply supported, $t = 2$

Mesh (nel)		3	12	48	192	Solution
Energy	WT (CRB1)	298.80969	393.25690	419.89374	426.73833	429.03071
	WT (CRB2)	336.93226	412.18318	425.41230	428.17104	
	PWu ($\alpha = 0.0062$)	429.7022	430.3069	429.4380	429.1430	
	PTL ($\alpha = 0.060$)	429.3074	436.3382	431.4247	429.6708	
	CHWu ($\alpha_1 = 0.0063, \alpha_2 = 0.99$)	428.4440	430.0680	429.3766	429.1276	
	CHTL ($\alpha_1 = 0.062, \alpha_2 = 0.99$)	429.4671	436.4756	431.4661	429.6816	
ω_c	WT (CRB1)	41.22420	45.52658	46.59817	46.86700	46.95656
	WT (CRB2)	44.35011	46.85908	46.97144	46.96275	
	PWu ($\alpha = 0.0062$)	33.8168	43.2970	46.0481	46.7303	
	PTL ($\alpha = 0.060$)	41.5424	45.5541	46.6082	46.8696	
	CHWu ($\alpha_1 = 0.0063, \alpha_2 = 0.99$)	34.3102	43.4254	46.0839	46.7393	
	CHTL ($\alpha_1 = 0.062, \alpha_2 = 0.99$)	42.1106	45.6931	46.6440	46.8786	
$-M_c$	PWu ($\alpha = 0.0062$)	3.7792	4.8245	5.0720	5.1351	
	PTL ($\alpha = 0.060$)	4.5727	5.0574	5.1321	5.1504	
	CHWu ($\alpha_1 = 0.0063, \alpha_2 = 0.99$)	3.9805	4.8434	5.0828	5.1381	
	CHTL ($\alpha_1 = 0.062, \alpha_2 = 0.99$)	4.8432	5.1040	5.1442	5.1534	

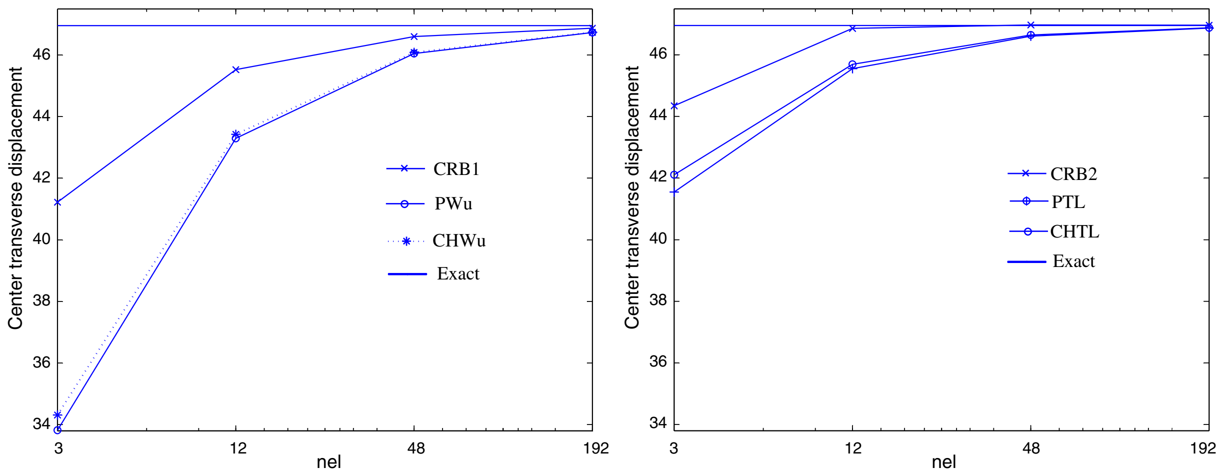

Fig. 11. Convergence study for a thick circular plate; simply supported, uniform load.

All six elements yield nearly identical results. when four-element mesh is used, CRB2 gets 99.78% of the center transverse displacement and 96.27% of energy obtained for 256 elements, CHWu gets 92.90% of center transverse displacement and 100.21% of the energy obtained for 256 elements, while CHTL gets 97.47% of the center transverse displacement and 101.58% of energy obtained for 256 elements.

4.3. Mesh distortion

To study the sensitivity to mesh distortion, a rough mesh model is used. Only 4 elements are used to model on quadrant of a clamped square plate. The center node of the mesh is moved along the main diagonal of the plate as shown in Fig. 12. Results are summarized in Table 11, and shown in Fig. 14.

Results indicate none of these elements is shear-locking, what's more, CRB1 and CHWu shows perfect feasibility.

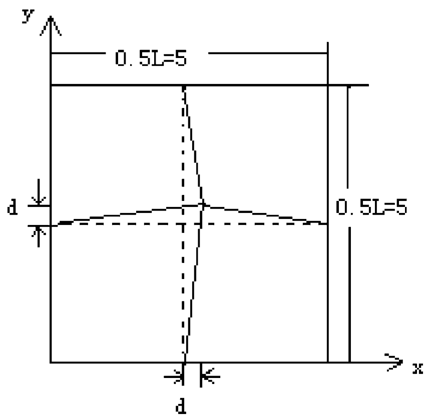

Fig. 12. Mesh distortion, symmetric.

Table 11
Mesh distortion—square plate, clamped, $t = 0.01$

d	WT (CRB1)	WT (CRB2)	PWu ($\alpha = 0.080$)	PTL ($\alpha = 0.31$)	CHWu ($\alpha_1 = 0.10, \alpha_2 = 0.70$)	CHTL ($\alpha_1 = 0.10, \alpha_2 = 0.70$)
−1.25	13.81498	24.22608	10.2150	6.6547	10.9332	6.9509
−1	13.90309	19.35489	11.0621	7.9741	11.7502	8.3366
−0.5	12.46806	12.84389	11.9249	10.3279	12.5725	10.8308
0.0	12.11830	12.11691	12.0221	11.7690	12.6097	12.2961
0.5	13.47277	13.31214	11.6300	12.1236	12.1729	12.5922
1	13.43153	16.46809	10.8854	11.5479	11.4627	12.0049
1.25	12.62252	19.47493	10.2970	10.8780	11.0035	11.4419

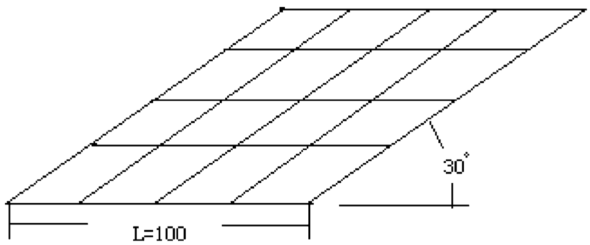

Fig. 13. Rhombic plate mesh (16 elements).

4.4. Highly skewed rhombic plates

A simply supported rhombic plate of side $L = 100$, material properties, $E = 10e+6$, $v = 0.3$ and $t = 1.0$, is loaded by a unit uniform loading.a 4×4 mesh used is shown in Fig. 13. Results are summarized in Table 12, and shown in Fig. 15. A comparison solution of 0.04455 has been obtained by Morley (see [15]).

For this problem, CRB1 has the best performance, which obtains 93.15% of the exact center transverse displacement for 16 elements, however when it comes to energy norm, which only obtain 84.55% of energy obtained for 256 elements. Among these new elements, CHWu perform best, which yields 88.66% of analyze center transverse displacement, 97.70% of energy obtained for 256 elements.

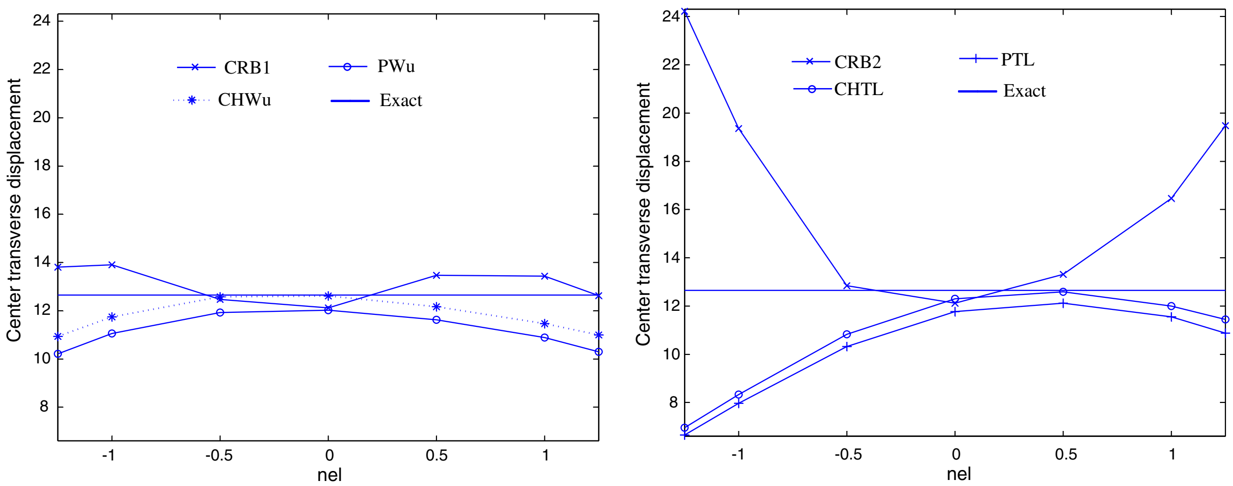

Fig. 14. Sensitivity to mesh distortion.

Table 12
Rhombic plate—uniform load SS1, $t = 1$, $L = 100$, $\theta = 30$

Mesh size		2 × 2	4 × 4	8 × 8	16 × 16	Solution
Energy	WT (CRB1)	50.38521	64.76201	72.32353	76.59302	
	WT (CRB2)	89.28845	72.14388	76.54601	79.00726	
	PWu ($\alpha = 0.00070$)	74.1712	74.1736	74.4676	76.7688	
	PTL ($\alpha = 0.055$)	76.7299	67.2454	72.6597	76.3584	
	CHWu ($\alpha_1 = 0.00070, \alpha_2 = 0.99$)	74.2280	77.5258	79.0335	79.35086	
	CHTL ($\alpha_1 = 0.055, \alpha_2 = 0.99$)	77.8365	69.4350	75.7482	78.6066	
ω_c	WT (CRB1)	0.04031	0.04150	0.04304	0.04446	0.044554
	WT (CRB2)	0.07143	0.04724	0.04538	0.04620	
	PWu ($\alpha = 0.00070$)	0.0193	0.0427	0.0439	0.0447	
	PTL ($\alpha = 0.055$)	0.0354	0.0381	0.0423	0.0443	
	CHWu ($\alpha_1 = 0.00070, \alpha_2 = 0.99$)	0.0194	0.0460	0.0479	0.0466	
	CHTL ($\alpha_1 = 0.055, \alpha_2 = 0.99$)	0.0360	0.0395	0.0444	0.0459	

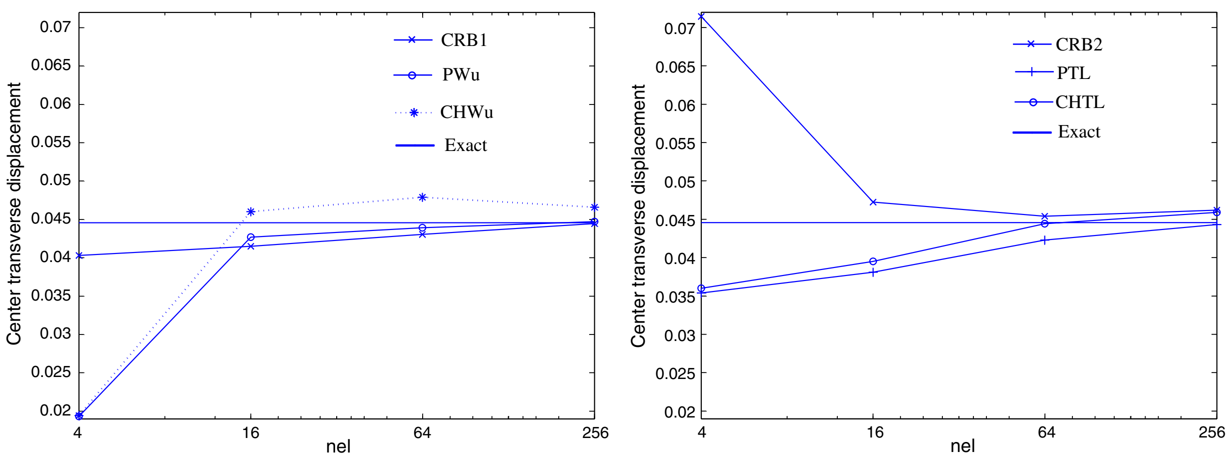

Fig. 15. Convergence study for a rhombic plate.

5. Concluding remarks

This paper has introduced two kinds of combined hybrid variational formulations for plate bending finite elements based upon the Reissner–Mindlin theory, according to whether assumed constant moment stress was introduced when assumed constant shear stress has been introduced. Due to two different options of incompatible displacement modes, four types of combined hybrid elements were proposed.

New elements, based on combined hybrid method, obtained more accurate energy than both CRB1 and CRB2 by changing combined parameter . Four elements showed good performances for a set of problems selected from the literature. Results show the introduction of assumed moment stress, on basis of assumed shear stress was introduced, is beneficial to improve the accuracy of center transverse displacement and moment stress. Especially, CHWu yielded excellent results. The mesh distortion test introduced in this paper shows the new formulations to be free of shear locking.

The remaining open question is how to choose the assumed shear stress, moment stress and incompatible modes. Numerical experiments show the resulting element performance is heavily dependent upon the options of the incompatible modes. In this paper, assumed shear and moment stress was chose to be constant and performed good application, but we did not analyze the effect of different choices of assumed shear and moment stress on the accuracy. The question of optimal conditions for the assumed shear and moment stress modes and incompatible modes is left for future research.

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