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Asymptotic error expansions for hypersingular integrals

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Abstract This paper presents quadrature formulae for hypersingular integrals $\int_a^b \frac{g(x)}{|x-t|^{1+\alpha}}\mathrm{d}x$, where $a < t < b$ and $0 < \alpha \leq 1$. The asymptotic error estimates obtained by Euler–Maclaurin expansions show that, if $g(x)$ is $2m$ times differentiable on $[a, b]$, the order of convergence is $O(h^{2\mu})$ for $\alpha = 1$ and $O(h^{2\mu-\alpha})$ for $0 < \alpha < 1$, where μ is a positive integer determined by the integrand. The advantages of these formulae are as follows: (1) using the formulae to evaluate hypersingular integrals is straightforward without need of calculating any weight; (2) the quadratures only involve $g(x)$, but not its derivatives, which implies these formulae can be easily applied for solving corresponding hypersingular boundary integral equations in that $g(x)$ is unknown; (3) more precise quadratures can be obtained by the Richardson extrapolation. Numerical experiments in this paper verify the theoretical analysis.

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1 Introduction

In this paper, we consider the *hypersingular* integral

$$I(g) = f.p. \int_a^b \frac{g(x)}{|x-t|^{1+\alpha}} \mathrm{d}x, \quad 0 < \alpha \leq 1, \; t \in (a, b), \tag{1}$$

where $f.p.$ denotes the *Hadamard finite part* of the hypersingular integral [24]. Such hypersingular integrals frequently appear in the *boundary integral equations* (BIEs) for many engineering applications such as the calculation of stresses in elasticity problems [7, 24]; the crack problems in the fracture mechanics [5, 6, 17], the scattering problems of time-harmonic waves by a crack [9, 20, 21]. Since the hypersingular integral (1) contributes to the dominant terms of the coefficient matrix of the related *boundary element method* (BEM), the success of BEM pivots on the accurate numerical evaluation of (1).

So far the weakly singular or Cauchy type singular integrals and singular integral equations have been intensively investigated. The related numerical methods and theoretical analyses can be found in [2, 3, 10, 11, 18, 19]. While for the hypersingular integrals, most of the numerical methods that have been proposed are for evaluating (1) with a second-order singularity, that is, $\alpha = 1$, particularly. In [16], Kabir et al. used the piecewise quadratic polynomial technique to solve hypersingular integral equations. In [15], Hui and Shia presented a Gaussian quadrature formula for (1), where the classical orthonormal polynomials such as Legendre and Chebyshev polynomials were used. Using quadrature rules based on interpolatory trigonometric polynomials, Kim and Choi [22] gave two quadrature formulae for evaluating (1), in which the cosine transformation and trigonometric polynomial interpolation at the practical abscissas were used, and a three-term recurrence relation was used to evaluate the quadrature weights. In [8], the parametric sigmoidal transformation is used to improve the asymptotic behavior of the Euler–Maclaurin formulae to deal with the second-order singularity, but the quadrature contains derivatives of the sigmoidal transformation. In [4], based on Brandão's approach to finite part integrals, Carley presented a method for the design of numerical quadrature rules that evaluate hypersingular integrals without requiring a detailed analysis of the integrand. The method can be regarded as a simplification and extension of the one in [23].

In this paper, we will present quadrature formulae to evaluate (1) for $0 < \alpha \leq 1$, which are obtained from the *Euler–Maclaurin expansions* for the error of the trapezoidal rule. This technique has been used to design quadratures for Cauchy singular and weakly singular integrals. Navot [31] constructed modified trapezoidal rules of integrals with algebraic and logarithmic singularities at the end point, and gave the Euler–Maclaurin expansions for the error. In [30], based on Mellin transform, Monegato and Lyness presented a unified approach for deriving the one-dimensional Euler–Maclaurin expansions for quadrature error functions defined on a finite interval when the integrand function has an algebraic singularity at one, or both endpoints. In [27, 28], Lyness studied the Euler–Maclaurin expansion technique for the evaluation of Cauchy principal integrals. In that paper, the integral was split into two parts. One can be calculated analytically and the other was evaluated by the trapezoidal rule with classical Euler–Maclaurin expansion. It's worth mentioning that Elliott and Venturino extended the Lyness' study to hypersingular integrals, and they introduced a sigmoidal transformation to make the integrand have certain periodicity [12]. However, their quadrature involves the derivatives of the density function ($g(x)$ in the hypersingular integral (1)), which makes it not easy to be applied for solving the boundary integral equations where the density function is unknown. Based on Euler–Maclaurin expansions of modified trapezoidal rule approximation, Sidi and Israeli [33] presented numerical quadrature methods for integrals of periodic functions with algebraic, logarithmic, and Cauchy singularities at the interior points of the interval. In [35], Xu and Zhao presented an extrapolation method for the *Nyström* scheme based on the quadrature formula in [33] to solve the BIEs of the second kind with periodic logarithmic kernels. The method is analyzed under the assumption that the inverse matrix of discrete equations exists and is uniformly bounded. In [14], Huang and Wang discussed extrapolation algorithms for solving the same type of BIEs by the *mechanical quadrature method* and analyzed the method in a more general setting.

The main purpose of this paper is to extend the quadrature method presented in [33] to the hypersingular integral (1). Based on a modified trapezoidal formula and the Euler–Maclaurin expansions for the error, several accurate quadrature formulae are constructed. Both theoretical and numerical results show that if $g(x)$ is $2m$ times differentiable on $[a, b]$, the accuracy order of the algorithms is $O(h^{2\mu})$ for $\alpha = 1$ and $O(h^{2\mu-\alpha})$ for $0 < \alpha < 1$. Here μ is a positive integer determined by the integrand. More accurate results can be achieved, especially, for the hypersingular integration with a periodic integrand. Furthermore, the accuracy order can be greatly improved by applying the extrapolation method [2, 25, 35, 37]. At the same time, this method can be easily extended to evaluate hypersigular integrals with a higher order singularity. The quadratures only use values of $g(x)$, but do not depend on its derivatives, which makes them easy to use in solving the hypersingular boundary integral equation. This also makes the application of the quadratures very straightforward and no detailed analysis of the integrand is needed.

This paper is organized as follows: in Section 2, we present quadrature formulae for hypersingular integrals and their asymptotic error expansions; in Section 3, we establish the extrapolation methods for hypersingular integrals with either periodic integrand or non-periodic integrand respectively; in Section 4, five numerical examples are displayed, in particular, Example 5 exhibits the application of these quadrature formulae in the context of the boundary integral equation.

2 Euler–Maclaurin expansions for hypersingular integrals

In this section, we will derive the quadrature formulae for hypersingular integrals (1) with $\alpha = 1$ and $0 < \alpha < 1$ respectively. The corresponding asymptotic errors are obtained based on the Euler–Maclaurin expansions.

To simplify notations, the double prime on a summation means that coefficients of the first and last terms in the summation are 1/2, that is, we let

$$\sum_{j=n_1}^{n_2}{}'' \omega_j = \frac{1}{2}\omega_{n_1} + \omega_{n_1+1} + \cdots + \omega_{n_2-1} + \frac{1}{2}\omega_{n_2}. \tag{2}$$

2.1 Quadrature formulae and their asymptotic expansions for $\alpha = 1$

From the definition of hypersingular integrals [24], one can write

$$f.p.\int_a^b \frac{g(x)}{(x-t)^2}\mathrm{d}x = \lim_{\varepsilon\to 0}\left[\int_a^{t-\varepsilon} \frac{g(x)}{(x-t)^2}\mathrm{d}x + \int_{t+\varepsilon}^b \frac{g(x)}{(x-t)^2}\mathrm{d}x - \frac{2g(t)}{\varepsilon}\right], \tag{3}$$

where $t \in (a, b)$. Let n be the number of equally spaced nodes used in the quadrature, $h = (b-a)/n$, $x_j = a + jh$ ($j = 0, 1, ..., n$) and *assume the singular point $x = t$ is one of the abscissas, that is, $t \in \{x_j\}_{j=1}^{n-1}$*. Using the classic Euler–Maclaurin expansion on a certain modified trapezoidal formula, we can derive the following quadrature formula.

Theorem 1 *Let $g(x)$ be $2m$ times differentiable on $[a, b]$, $G(x) = \frac{g(x)}{(x-t)^2}$ and $I(g) = \int_a^b G(x)\mathrm{d}x$, the quadrature formula is*

$$Q(h) = h\sum_{j=0, x_j\neq t}^{n}{}'' \frac{g(x_j) - g(t)}{(x_j - t)^2} - \left(\frac{1}{b-t} + \frac{1}{t-a}\right)g(t)$$
$$-\sum_{\mu=1}^{m-1}\left[\frac{B_{2\mu}g(t)h^{2\mu}}{(t-a)^{2\mu+1}} + \frac{B_{2\mu}g(t)h^{2\mu}}{(b-t)^{2\mu+1}}\right], \tag{4}$$

where $B_{2\mu}$ is the Bernoulli numbers. Then its asymptotic error estimate is

$$E_n(h) = I(g) - Q(h) = \sum_{\mu=1}^{m-1}\frac{B_{2\mu}}{(2\mu)!}\left[G^{(2\mu-1)}(a) - G^{(2\mu-1)}(b)\right]h^{2\mu}$$
$$+\frac{hg''(t)}{2} + O(h^{2m}). \tag{5}$$

Proof To derive the quadrature, first separate the integrand and obtain

$$f.p.\int_a^b \frac{g(x)}{(x-t)^2}\mathrm{d}x = f.p.\int_a^b \frac{g(t)}{(x-t)^2}\mathrm{d}x + p.v.\int_a^b \frac{g(x)-g(t)}{(x-t)^2}\mathrm{d}x = I_1(g)+I_2(g), \tag{6}$$

where $p.v.$ denotes *Cauchy principal value*. Based on (3), we have

$$I_1(g) = f.p.\int_a^b \frac{g(t)}{(x-t)^2}\mathrm{d}x = -\left(\frac{1}{b-t}+\frac{1}{t-a}\right)g(t). \tag{7}$$

Without loss of generality, we assume that $t-a \le b-t$. Since $t \in \{x_j\}_{j=1}^{n-1}$, we set $\tilde{b} = 2t-a$, so that t is the midpoint of interval $[a, \tilde{b}]$. Then we decompose

$$\begin{aligned} I_2(g) &= p.v.\int_a^b \frac{g(x)-g(t)}{(x-t)^2}\mathrm{d}x \\ &= p.v.\int_a^{\tilde{b}} \frac{g(x)-g(t)}{(x-t)^2}\mathrm{d}x + \int_{\tilde{b}}^b \frac{g(x)-g(t)}{(x-t)^2}\mathrm{d}x \\ &= I_{21}(g)+I_{22}(g), \end{aligned} \tag{8}$$

where $I_{22}(g)$ is a proper integral. By the Euler–Maclaurin formula, we have

$$\begin{aligned} E_{22}(h) &= I_{22}(g) - h\sum_{x_j\ge\tilde{b}}{}'' \frac{g(x_j)-g(t)}{(x_j-t)^2} \\ &= \sum_{\mu=1}^{m-1}\frac{B_{2\mu}h^{2\mu}}{(2\mu)!}\left[G^{(2\mu-1)}(\tilde{b}) - G^{(2\mu-1)}(b)\right] \\ &\quad + \sum_{\mu=1}^{m-1} B_{2\mu}h^{2\mu}g(t)\left[\frac{1}{(\tilde{b}-t)^{2\mu+1}} - \frac{1}{(b-t)^{2\mu+1}}\right] + O(h^{2m}). \end{aligned} \tag{9}$$

Now consider $I_{21}(g)$. Since

$$\sum_{x_j\le\tilde{b},t\ne x_j}(x_j-t)^{-1} = 0, \text{ and } p.v.\int_a^{\tilde{b}} \frac{g'(t)}{x-t}\mathrm{d}x = 0,$$

we have

$$I_{21}(g) = p.v\int_a^{\tilde{b}} \frac{g(x)-g(t)}{(x-t)^2}\mathrm{d}x = \int_a^{\tilde{b}} \frac{g(x)-g(t)-g'(t)(x-t)}{(x-t)^2}\mathrm{d}x,$$

where $I_{21}(g)$ is a proper integral, and the integrand becomes $g''(t)/2$ as $x \to t$. Applying the Euler–Maclaurin formula, we get

$$
\begin{aligned}
E_{21}(h) &= p.v. \int_a^{\tilde{b}} \frac{g(x)-g(t)}{(x-t)^2}\mathrm{d}x - h \sum_{x_j \le \tilde{b}, t \neq x_j}{}^{\prime\prime} \frac{g(x_j)-g(t)-g'(t)(x_j-t)}{(x_j-t)^2} \\
&= \frac{hg''(t)}{2} + \sum_{\mu=1}^{m-1} \frac{B_{2\mu}h^{2\mu}}{(2\mu)!} \left\{ \frac{\partial^{2\mu-1}}{\partial x^{2\mu-1}} \left[\frac{g(x)-g(t)-g'(t)(x-t)}{(x-t)^2} \right] \Bigg|_{x=a} \right. \\
&\qquad \left. - \frac{\partial^{2\mu-1}}{\partial x^{2\mu-1}} \left[\frac{g(x)-g(t)-g'(t)(x-t)}{(x-t)^2} \right] \Bigg|_{x=\tilde{b}} \right\} \\
&\qquad + O(h^{2m}) \\
&= \frac{hg''(t)}{2} + E_{21}^{(1)}(h) + E_{21}^{(2)}(h) + E_{21}^{(3)}(h) + O(h^{2m}); \qquad (10)
\end{aligned}
$$

here

$$
E_{21}^{(1)}(h) = \sum_{\mu=1}^{m-1} \frac{B_{2\mu}h^{2\mu}}{(2\mu)!} \left[G^{(2\mu-1)}(a) - G^{(2\mu-1)}(\tilde{b}) \right], \qquad (11)
$$

$$
E_{21}^{(2)}(h) = \sum_{\mu=1}^{m-1} B_{2\mu}h^{2\mu} g(t) \left[\frac{1}{(a-t)^{2\mu+1}} - \frac{1}{(\tilde{b}-t)^{2\mu+1}} \right], \qquad (12)
$$

and

$$
E_{21}^{(3)}(h) = \sum_{\mu=1}^{m-1} \frac{B_{2\mu}h^{2\mu}}{(2\mu)!} g'(t) \left[\frac{1}{(a-t)^{2\mu}} - \frac{1}{(\tilde{b}-t)^{2\mu}} \right] = 0. \qquad (13)
$$

Adding (8)–(13) together, we get

$$
\begin{aligned}
E_2(h) &= I_2(g) - h \sum_{j=0, t \neq x_j}^{n}{}^{\prime\prime} \frac{g(x_j)-g(t)}{(x_j-t)^2} \\
&= \sum_{\mu=1}^{m-1} \frac{B_{2\mu}h^{2\mu}}{(2\mu)!} \left[G^{(2\mu-1)}(a) - G^{(2\mu-1)}(b) \right] \\
&\quad - \sum_{\mu=1}^{m-1} B_{2\mu}h^{2\mu} g(t) \left[\frac{1}{(t-a)^{2\mu+1}} + \frac{1}{(b-t)^{2\mu+1}} \right] + \frac{hg''(t)}{2} + O(h^{2m}). \qquad (14)
\end{aligned}
$$

This completes the proof. □

However, there is a term $hg''(t)/2$ in the asymptotic error expansion (5), which means that the order of error is only $O(h)$. By using the Richardson extrapolation, we can obtain a new quadrature rule with higher order of accuracy.

Corollary 1 *Under the assumptions of Theorem* 1, *the modified quadrature formula*

$$\bar{Q}(h) = h\sum_{j=1}^{n} \frac{g(x_{2j-1}) - g(t)}{(x_{2j-1} - t)^2} - g(t)\left(\frac{1}{b-t} + \frac{1}{t-a}\right)$$
$$- \sum_{\mu=1}^{m-1}(2^{1-2\mu} - 1)B_{2\mu}h^{2\mu}g(t)\left[\frac{1}{(t-a)^{2\mu+1}} + \frac{1}{(b-t)^{2\mu+1}}\right] \quad (15)$$

has the following asymptotic error expansion

$$E_n(h) = I(g) - \bar{Q}(h) = \sum_{\mu=1}^{m-1} \frac{(2^{1-2\mu} - 1)B_{2\mu}h^{2\mu}}{(2\mu)!}$$
$$\times\left[G^{(2\mu-1)}(a) - G^{(2\mu-1)}(b)\right] + O(h^{2m}). \quad (16)$$

Corollary 1 can be easily obtained from Theorem 1 by setting $\bar{Q}(h) = 2Q(h/2) - Q(h)$. The term of $hg''(t)/2$ is eliminated through the Richardson extrapolation. The corresponding error expansion shows that the accuracy order of the new quadrature, $\bar{Q}(h)$, is $O(h^2)$.

Remark 1 The same technique can be used to derive quadrature formulae for hypersingular integrals with higher order of singularity, $\alpha > 1$ in (1), or for the hypersingular integral with a more complicated integrand such as $\int_a^b \frac{g(x)\ln|x-t|}{|x-t|^{1+\alpha}}\mathrm{d}x$.

2.2 Quadrature formulae and their asymptotic expansions for $0 < \alpha < 1$

Next we consider the evaluation of hypersingular integrals (1) for $0 < \alpha < 1$. The quadrature formulae and corresponding asymptotic error expansions are presented. Based on [24], the definition of the finite part of such hypersingular integrals reads

$$f.p.\int_a^b \frac{g(x)}{|x-t|^{1+\alpha}}\mathrm{d}x = \lim_{\varepsilon\to 0}\left[\int_a^{t-\varepsilon} \frac{g(x)}{|x-t|^{1+\alpha}}\mathrm{d}x\right.$$
$$\left. + \int_{t+\varepsilon}^{b} \frac{g(x)}{|x-t|^{1+\alpha}}\mathrm{d}x - \frac{2g(t)}{\alpha\varepsilon^{\alpha}}\right], \quad (17)$$

in particular,

$$f.p.\int_a^b \frac{\mathrm{d}x}{|x-t|^{1+\alpha}} = -\frac{1}{\alpha}\left(\frac{1}{|t-a|^{\alpha}} + \frac{1}{|b-t|^{\alpha}}\right). \tag{18}$$

Lemma 1 ([26, 31, 33]) *Let $g(x)$ be $2m$ times differentiable on $[a, b]$, $G(x) = (x-a)^s g(x)$ with $s > -1$ and $I(g) = \int_a^b G(x)\mathrm{d}x$. Then the error of the composite trapezoidal rule is given by*

$$\begin{aligned} E_n(h) &= I(g) - h\sum_{j=1}^{n}{}'' G(x_j) \\ &= -\sum_{\mu=1}^{m-1}\frac{B_{2\mu}}{(2\mu)!}G^{(2\mu-1)}(b)h^{2\mu} \\ &\quad -\sum_{\mu=0}^{2m-1}\frac{\zeta(-s-\mu)}{\mu!}g^{(\mu)}(a)h^{\mu+s+1} + O(h^{2m}), \end{aligned} \tag{19}$$

where $\zeta(z)$ is the Riemann zeta function. If $g(x)$ is infinitely differentiable on $[a, b]$, then $E_n(h)$ has the following asymptotic expansion

$$E_n(h) \sim -\sum_{\mu=1}^{\infty}\frac{B_{2\mu}}{(2\mu)!}G^{(2\mu-1)}(b)h^{2\mu} - \sum_{\mu=0}^{\infty}\frac{\zeta(-s-\mu)}{\mu!}g^{(\mu)}(a)h^{\mu+s+1}. \tag{20}$$

Below we derive the quadrature formula for the hypersingular integrals (1) for $0 < \alpha < 1$ by utilizing the finite part of the hypersingular integrals (17) and Lemma 1.

Theorem 2 *Let $g(x)$ be $2m$ times differentiable on $[a, b]$, $G(x) = \frac{g(x)}{|x-t|^{1+\alpha}}$ with $0 < \alpha < 1$ and $I(g) = \int_a^b G(x)\mathrm{d}x$. The quadrature is*

$$\begin{aligned} Q(h) &= h\sum_{j=0,t\neq x_j}^{n}{}'' \frac{g(x_j)-g(t)}{|x_j-t|^{1+\alpha}} - \frac{g(t)}{\alpha}\left[\frac{1}{(t-a)^{\alpha}} + \frac{1}{(b-t)^{\alpha}}\right] \\ &\quad -\sum_{\mu=1}^{m-1}\frac{B_{2\mu}h^{2\mu}\varphi(2\mu-1,\alpha)g(t)}{(2\mu)!}\left[\frac{1}{(t-a)^{2\mu+\alpha}} + \frac{1}{(b-t)^{2\mu+\alpha}}\right] \end{aligned} \tag{21}$$

and $\varphi(\mu,\alpha)=\prod_{i=1}^{\mu}(i+\alpha)$. *Then the asymptotic error estimate is*

$$E_n(h)=I(g)-Q(h)=\sum_{\mu=1}^{m-1}\frac{B_{2\mu}h^{2\mu}}{(2\mu)!}\left[G^{(2\mu-1)}(a)-G^{(2\mu-1)}(b)\right]$$

$$-\sum_{\mu=1}^{m-1}\frac{2\zeta(\alpha-2\mu+1)g^{(2\mu)}(t)h^{2\mu-\alpha}}{(2\mu-1)!}+O(h^{2m}). \tag{22}$$

Proof From (17), we have

$$f.p.\int_a^b\frac{g(x)}{|x-t|^{1+\alpha}}\mathrm{d}x=\int_a^b\frac{g(x)-g(t)}{|x-t|^{1+\alpha}}\mathrm{d}x+f.p.\int_a^b\frac{g(t)}{|x-t|^{1+\alpha}}\mathrm{d}x$$

$$=\int_a^b\frac{g(x)-g(t)}{|x-t|^{1+\alpha}}\mathrm{d}x-\frac{g(t)}{\alpha}\left[\frac{1}{(t-a)^{\alpha}}+\frac{1}{(b-t)^{\alpha}}\right]. \tag{23}$$

Since $g(x)\in C^{2m}[a,b]$, $(g(x)-g(t))/|x-t|^{1+\alpha}$ is weakly singular. Without loss of generality, we assume that $x<t$ and $t\in\{x_j\}_{j=1}^{n-1}$. From Lemma 1 and $\int_0^{t-a}\frac{g(t-z)}{z^{1+\alpha}}\mathrm{d}z=\int_a^t G(x)\mathrm{d}x$, we have the following error estimates

$$E_1(h)=\int_a^t\frac{g(x)-g(t)}{|x-t|^{1+\alpha}}\mathrm{d}x-h{\sum_{x_j<t}}''\frac{g(x_j)-g(t)}{|x_j-t|^{1+\alpha}}$$

$$=\sum_{\mu=1}^{m-1}\frac{B_{2\mu}h^{2\mu}}{(2\mu)!}\left(\frac{g(x)-g(t)}{|x-t|^{1+\alpha}}\right)^{(2\mu-1)}\Bigg|_{x=a}$$

$$-\sum_{\mu=0}^{2m-1}(-1)^{\mu}\frac{\zeta(\alpha-\mu)}{\mu!}\left(\frac{g(x)-g(t)}{|x-t|}\right)^{(\mu)}\Bigg|_{x=t}h^{\mu-\alpha+1}+O(h^{2m})$$

$$=\sum_{\mu=1}^{m-1}\frac{B_{2\mu}h^{2\mu}}{(2\mu)!}G^{(2\mu-1)}(a)+\sum_{\mu=1}^{m-1}\frac{g(t)B_{2\mu}h^{2\mu}}{(2\mu)!}\left(\frac{1}{|x-t|^{1+\alpha}}\right)^{(2\mu-1)}\Bigg|_{x=a}$$

$$-\sum_{\mu=0}^{2m-1}(-1)^{\mu}\frac{\zeta(\alpha-\mu)}{\mu!}\left(-g'(t)\right)^{(\mu)}h^{\mu-\alpha+1}+O(h^{2m})$$

$$= \sum_{\mu=1}^{m-1} \frac{B_{2\mu} h^{2\mu}}{(2\mu)!} G^{(2\mu-1)}(a) - \sum_{\mu=1}^{m-1} \frac{g(t) B_{2\mu} h^{2\mu}}{(2\mu)!} \frac{\varphi(2\mu-1,\alpha)}{(t-a)^{2\mu+\alpha}}$$

$$+ \sum_{\mu=0}^{2m-1} (-1)^{\mu} \frac{\zeta(\alpha-\mu)}{\mu!} g^{(\mu+1)}(t) h^{\mu-\alpha+1} + O(h^{2m}), \tag{24}$$

and

$$E_2(h) = \int_t^b \frac{g(x)-g(t)}{|x-t|^{1+\alpha}} \mathrm{d}x - h {\sum_{x_j>t}}'' \frac{g(x_j)-g(t)}{|x_j-t|^{1+\alpha}}$$

$$= -\sum_{\mu=1}^{m-1} \frac{B_{2\mu} h^{2\mu}}{(2\mu)!} \left(\frac{g(x)-g(t)}{|x-t|^{1+\alpha}} \right)^{(2\mu-1)} \Bigg|_{x=b}$$

$$- \sum_{\mu=0}^{2m-1} \frac{\zeta(\alpha-\mu)}{\mu!} \left(\frac{g(x)-g(t)}{|x-t|} \right)^{(\mu)} \Bigg|_{x=t} h^{\mu-\alpha+1} + O(h^{2m})$$

$$= -\sum_{\mu=1}^{m-1} \frac{B_{2\mu} h^{2\mu}}{(2\mu)!} G^{(2\mu-1)}(b) + \sum_{\mu=1}^{m-1} \frac{g(t) B_{2\mu} h^{2\mu}}{(2\mu)!} \left(\frac{1}{|x-t|^{1+\alpha}} \right)^{(2\mu-1)} \Bigg|_{x=b}$$

$$- \sum_{\mu=0}^{2m-1} \frac{\zeta(\alpha-\mu)}{\mu!} \left(g'(t) \right)^{(\mu)} h^{\mu-\alpha+1} + O(h^{2m})$$

$$= -\sum_{\mu=1}^{m-1} \frac{B_{2\mu} h^{2\mu}}{(2\mu)!} G^{(2\mu-1)}(b) - \sum_{\mu=1}^{m-1} \frac{g(t) B_{2\mu} h^{2\mu}}{(2\mu)!} \frac{\varphi(2\mu-1,\alpha)}{(b-t)^{2\mu+\alpha}}$$

$$- \sum_{\mu=0}^{2m-1} \frac{\zeta(\alpha-\mu)}{\mu!} g^{(\mu+1)}(t) h^{\mu-\alpha+1} + O(h^{2m}). \tag{25}$$

Putting (24) and (25) together, we derive

$$E_1(h) + E_2(h) = \sum_{\mu=1}^{m-1} \frac{B_{2\mu} h^{2\mu}}{(2\mu)!} \left[G^{(2\mu-1)}(a) - G^{(2\mu-1)}(b) \right]$$

$$- \sum_{\mu=1}^{m-1} \frac{g(t) B_{2\mu} h^{2\mu} \varphi(2\mu-1,\alpha)}{(2\mu)!} \left[\frac{1}{(t-a)^{2\mu+\alpha}} + \frac{1}{(b-t)^{2\mu+\alpha}} \right]$$

$$-\sum_{\mu=1}^{m-1}\frac{2\zeta(\alpha-2\mu+1)g^{(2\mu)}(t)h^{2\mu-\alpha}}{(2\mu-1)!}+O(h^{2m}). \tag{26}$$

Combining (23) and (26), we complete the proof of Theorem 2. □

3 Quadrature formulae based on the Richardson extrapolation

In this section, we will derive several quadrature formulae with higher order of accuracy by the Richardson extrapolation. As seen from (16) and (22), $E_n(h)$ contains terms involving $G^{2\mu-1}(a) - G^{2\mu-1}(b)$. If $G(x)$ is a periodic function on $[a, b]$, those related terms will vanish. Then quadrature formulae with higher order of accuracy can be achieved directly by applying the Richardson extrapolation. However, if $G(x)$ is not periodic, we need to use the extrapolation in a different way. Then highly accurate results can still be obtained. We will discuss these two cases separately.

Theorem 3 *Assume that the function* $g(x)$ *is* $2m$ *times differentiable on* $[a, b]$ *and* $G(x)$ *is a periodic function with the period* $T = b - a$. *Moreover, if* $G(x)$ *is also* $2m$ *times differentiable on* $(-\infty, +\infty)\backslash\{t + kT\}_{k=-\infty}^{k=+\infty}$, *the following conclusions hold:*

(a) for $G(x) = g(x)/(x - t)^2$, *by* (16) *there exists the error estimate*

$$E_n(h) = I(g) - Q(h) = O(h^{2m}); \tag{27}$$

(b) for $G(x) = g(x)/|x - t|^{1+\alpha}$ $(0 < \alpha < 1)$, *by* (22) *the error has the asymptotic expansion*

$$E_n(h)=I(g)-Q(h)=-\sum_{\mu=1}^{m-1}\frac{2\zeta(\alpha-2\mu+1)g^{(2\mu)}(t)h^{2\mu-\alpha}}{(2\mu-1)!}+O(h^{2m}); \tag{28}$$

In case of (b), based on the asymptotic expansion (28), we derive the following quadrature formulae with higher order of accuracy by the Richardson extrapolations.

Theorem 4 *Under the assumption of Theorem* 3 *with* $Q(h)$ *defined as in* (21), *for* $G(x) = \frac{g(x)}{|x-t|^{1+\alpha}}$ $(0 < \alpha < 1)$, *the* K*-th Richardson extrapolation of* (21) *is implemented by*

$$\begin{cases} Q^{(0)}(h) = Q(h), \\ Q^{(\mathrm{K})}(h) = \left[2^{2\mathrm{K}-\alpha}Q^{(\mathrm{K}-1)}(h/2) - Q^{(\mathrm{K}-1)}(h)\right] / \left(2^{2\mathrm{K}-\alpha}-1\right),\ 1 \le \mathrm{K} \le m-1, \end{cases} \tag{29}$$

and the error of $Q^{(\mathrm{K})}(h)$ *has the following asymptotic expansion*

$$E_n^{(\mathrm{K})}(h) = \sum_{\mu=\mathrm{K}+1}^{m-1} A_\mu^{(\mathrm{K})} h^{2\mu-\alpha} + O(h^{2m}), \tag{30}$$

where $A_\mu^{(\mathrm{K})} = \left(2^{2\mathrm{K}-2\mu} - 1\right) A_\mu^{(\mathrm{K}-1)} / \left(2^{2\mathrm{K}-\alpha} - 1\right)$ *with* $A_\mu^{(0)} = -2\zeta(\alpha - 2\mu + 1) g^{(2\mu)}(t) / (2\mu - 1)!$ *and* $\mu \geq \mathrm{K} + 1$.

Proof Using mathematical induction, we can derive (29) and (30). □

If $G(x)$ is not periodic on $[a, b]$, we can implement the Richardson extrapolation in the following way.

Theorem 5 *Under the assumptions of Theorem* 1 *and letting* $\bar{Q}(h)$ *be defined as* (15), *the* K-*th Richardson extrapolation gives*

$$\begin{cases} \bar{Q}^{(0)}(h) = \bar{Q}(h) \\ \bar{Q}^{(\mathrm{K})}(h) = \left[2^{2\mathrm{K}} \bar{Q}^{(\mathrm{K}-1)}(h/2) - \bar{Q}^{(\mathrm{K}-1)}(h)\right] / \left(2^{2\mathrm{K}} - 1\right), \ 1 \leq \mathrm{K} \leq m - 1, \end{cases} \tag{31}$$

and there holds the following error asymptotic expansion

$$E_n^{(\mathrm{K})}(g) = \sum_{\mu=\mathrm{K}+1}^{m-1} c_\mu^{(\mathrm{K})} h^{2\mu} + O(h^{2m}), \tag{32}$$

where $E_n^{(\mathrm{K})}(h) = I(g) - \bar{Q}^{(\mathrm{K})}(h)$, $C_\mu^{(0)} = \frac{B_{2\mu}(2^{1-2\mu}-1)[G^{(2\mu-1)}(a) - G^{(2\mu-1)}(b)]}{(2\mu)!}$, $\mu = 1, \ldots,$ $m - 1$, $C_\mu^{(\mathrm{K})} = (\frac{2^{2\mathrm{K}}}{2^{2\mu}} - 1) C_\mu^{(\mathrm{K}-1)}$.

It is easy to prove (31) and (32).

Theorem 6 *Under the assumptions of Theorem* 2 *with* $Q(h)$ *defined as in* (21), *the quadrature after* K-*th Richardson extrapolation* $\bar{Q}^{(\mathrm{K})}(h)$ *reads*

$$\begin{cases} \bar{Q}^{(0)}(h) = Q(h) \\ \widetilde{Q}^{(\mathrm{K})}(h) = \left[2^{2\mathrm{K}-\alpha} \bar{Q}^{(\mathrm{K}-1)}(h/2) - \bar{Q}^{(\mathrm{K}-1)}(h)\right] / \left(2^{2\mathrm{K}-\alpha} - 1\right), \\ \bar{Q}^{(\mathrm{K})}(h) = \left[2^{2\mathrm{K}} \widetilde{Q}^{(\mathrm{K}-1)}(h/2) - \widetilde{Q}^{(\mathrm{K}-1)}(h)\right] / \left(2^{2\mathrm{K}} - 1\right), \ 1 \leq \mathrm{K} \leq m - 1, \end{cases} \tag{33}$$

Proof To simplify the notations, we let $\bar{A}_\mu^{(0)} = \frac{B_{2\mu}}{(2\mu)!}\left[G^{(2\mu-1)}(a) - G^{(2\mu-1)}(b)\right]$, $\bar{C}_\mu^{(0)} = -\frac{2\zeta(\alpha-2\mu+1)g^{(2\mu)}(t)}{(2\mu-1)!}$, and then rewrite (21) as

$$I(g) = \bar{Q}^{(0)}(h) + \sum_{\mu=1}^{m-1} \bar{A}_\mu^{(0)} h^{2\mu} + \sum_{\mu=1}^{m-1} \bar{C}_\mu^{(0)} h^{2\mu-\alpha} + O(h^{2m}), \tag{34}$$

where $\bar{Q}^{(0)}(h) = Q(h)$.

Step 1: Replacing h in (34) by $h/2$, we have

$$I(g) = \bar{Q}^{(0)}\left(\frac{h}{2}\right) + \sum_{\mu=1}^{m-1} \bar{A}_{\mu}^{(0)}\left(\frac{h}{2}\right)^{2\mu} + \sum_{\mu=1}^{m-1} \bar{C}_{\mu}^{(0)}\left(\frac{h}{2}\right)^{2\mu-\alpha} + O(h^{2m}). \quad (35)$$

Multiplying (35) by $2^{2-\alpha}$, then subtracting (34) from the result and finally dividing both sides by $2^{2-\alpha} - 1$, we obtain

$$\widetilde{Q}^{(0)}(h) = I(g) - \sum_{\mu=1}^{m-1} \widetilde{A}_{\mu}^{(0)} h^{2\mu} - \sum_{\mu=2}^{m-1} \widetilde{C}_{\mu}^{(0)} h^{2\mu-\alpha} + O(h^{2m}), \quad (36)$$

where $\widetilde{Q}^{(0)}(h) = [2^{2-\alpha}\bar{Q}^{(0)}(h/2) - \bar{Q}^{(0)}(h)]/[2^{2-\alpha} - 1]$, $\widetilde{A}_{\mu}^{(0)} = [2^{2-\alpha}/2^{2(\mu-1)} - 1]/[2^{2-\alpha} - 1]\bar{A}_{\mu}^{(0)}$ $(\mu = 1, ..., m-1)$ and $\widetilde{C}_{\mu}^{(0)} = [2^{-2(\mu-1)} - 1]/[2^{2-\alpha} - 1]\bar{C}_{\mu}^{(0)}$ $(\mu = 2, ..., m-1)$.

Step 2: Also replacing h in (36) by $h/2$, we have

$$I(g) = \widetilde{Q}^{(0)}\left(\frac{h}{2}\right) + \sum_{\mu=1}^{m-1} \widetilde{A}_{\mu}^{(0)}\left(\frac{h}{2}\right)^{2\mu} + \sum_{\mu=2}^{m-1} \widetilde{C}_{\mu}^{(0)}\left(\frac{h}{2}\right)^{2\mu-\alpha} + O(h^{2m}). \quad (37)$$

Multiplying (37) by 2^2, then subtracting (36) from the result and finally dividing both sides by $2^2 - 1$, we get

$$\bar{Q}^{(1)}(h) = I(g) - \sum_{\mu=2}^{m-1} \bar{A}_{\mu}^{(1)} h^{2\mu} - \sum_{\mu=2}^{m-1} \bar{C}_{\mu}^{(1)} h^{2\mu-\alpha} + O(h^{2m}), \quad (38)$$

where $\bar{Q}^{(1)}(h) = [2^2\widetilde{Q}^{(0)}(h/2) - \widetilde{Q}^{(0)}(h)]/[2^2 - 1]$, $\bar{A}_{\mu}^{(1)} = [2^2/2^{2(\mu-1)} - 1]/[2^2 - 1]\widetilde{A}_{\mu}^{(0)}$ and $\bar{C}_{\mu}^{(1)} = [2^2/2^{2(\mu-1)-\alpha} - 1]/[2^2 - 1]\widetilde{C}_{\mu}^{(0)}$ $(\mu = 2, ..., m-1)$.

Repeating the above steps, we can get the K-th Richardson extrapolation formulae for (21). □

In practice, it's usually enough to use the first or second Richardson extrapolation formula and get the desired accuracy.

4 Numerical experiments

In this section, we will present several tests to verify the error analysis given in last section numerically. The numerical results for the following kinds of hypersingular integrals are given: (a) non-periodic hypersingular integrals, (b) periodic hypersingular integrals by periodization, (c) periodic hypersingular integrals.

Example 1 Calculate the hypersingular integral

$$(I(g))(y) = \int_0^1 \frac{g(x)}{(x-y)^2}\mathrm{d}x,\ g(x) = (2x-1)^3, \tag{39}$$

where the exact solution is

$$(I(g))(y) = 8(2y-1) + 6(2y-1)^2 \log\left[(1-y)/y\right] - \frac{(2y-1)^3}{y(1-y)}.$$

Since $G(x) = \frac{(2x-1)^3}{(x-y)^2}$ is not periodic on [0, 1], we use the formulae (15) and (31). The numerical results are listed in Tables 1–2, where the number of nodes $n = \frac{1}{h}$, $e_n^{(\mathrm{K})}$ (K = 0, 1, 2) are the absolute errors of the K-th Richardson extrapolation, and $r_{2n}^{(\mathrm{K})} = e_n^{(\mathrm{K})}/e_{2n}^{(\mathrm{K})}$. Clearly, the numerical results in both tables show that

$$r_n^{(\mathrm{K})} \approx 2^{2\mathrm{K}+2}, \mathrm{K} = 0, 1, 2,$$

which verifies (16) and (32) perfectly.

Remark 2 As the value of y is close to the endpoint, the accuracy of numerical solutions decreases but the rate of convergence doesn't change. For example, it's seen than in Table 2, as quadrature (15) is applied, the error $e_n^{(0)} = 1.983 \times 10^{-2}$ as the number of abscissas $n = 2^{10}$ but $r_n^{(0)} = 2$, which still matches the analysis (16). Furthermore, the accuracy is improved by using quadratures after extrapolations (31). In Table 2, $e_n^{(2)} = 7.604 \times 10^{-8}$ as $n = 2^{10}$ after the second Richardson extrapolation.

Remark 3 The derivation of quadratures presented in this paper requires that the singular point be one of the equally spaced abscissas. In practice, the efficiency of numerical integrals can be improved by the adaptive mesh

Table 1 Numerical results based on the quadratures (15) and (31) for the hypersingular integral $\int_0^1 \frac{(2x-1)^3}{(x-y)^2}\mathrm{d}x$ at $y = \frac{1}{4}$

$n = \frac{1}{h}$	2^3	2^4	2^5	2^6	2^7	2^8	2^9
$e_n^{(0)}$	2.317e-2	6.074e-3	1.537e-3	3.854e-4	9.642e-5	2.411e-5	6.028e-6
$r_n^{(0)}$		$2^{1.931}$	$2^{1.982}$	$2^{1.995}$	$2^{1.998}$	$2^{1.999}$	$2^{1.999}$
$e_n^{(1)}$		3.751e-4	2.443e-5	1.546e-6	9.696e-8	6.065e-9	3.790e-10
$r_n^{(1)}$			$2^{3.940}$	$2^{3.982}$	$2^{3.995}$	$2^{3.998}$	$2^{3.999}$
$e_n^{(2)}$			1.058e-6	2.022e-8	3.360e-10	5.000e-12	7.818e-14
$r_n^{(2)}$				$2^{5.709}$	$2^{5.912}$	$2^{5.980}$	$2^{5.999}$

Note that the rates of convergence in errors match the theoretical analysis (16) and (32) respectively

Table 2 Numerical results based on the quadratures (15) and (31) for the hypersingular integral $\int_0^1 \frac{(2x-1)^3}{(x-y)^2}\mathrm{d}x$ at $y = \frac{1}{64}$

$n = \frac{1}{h}$	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
$e_n^{(0)}$	3.357e-0	1.162e-0	3.109e-1	7.900e-2	1.983e-2	4.962e-3	1.240e-3
$r_n^{(0)}$		$2^{1.530}$	$2^{1.902}$	$2^{1.976}$	$2^{1.994}$	$2^{1.998}$	$2^{1.999}$
$e_n^{(1)}$		4.309e-1	2.707e-2	1.695e-3	1.060e-4	6.630e-6	4.143e-7
$r_n^{(1)}$			$2^{3.992}$	$2^{3.996}$	$2^{3.999}$	$2^{3.999}$	$2^{3.999}$
$e_n^{(2)}$			1.502e-4	3.980e-6	7.604e-8	1.262e-9	1.900e-11
$r_n^{(2)}$				$2^{5.238}$	$2^{5.710}$	$2^{5.913}$	$2^{6.050}$

Note that the rates of convergence in errors match the theoretical analysis (16) and (32) respectively

method. For example, utilize (31) on a small interval that contains y and let y be one of the nodes for the quadrature, while applying the adaptive Simpson's rule on the rest interval. Since asymptotic error estimates are known, it is easy to match errors on both intervals.

Remark 4 As stated in Theorem 3, the quadrature for integral $I(g) = \int_a^b G(x)dx$ has higher order of convergence rate if $G(x)$ is periodic. Applying quadratures (15) on the periodizated integrand will improve the accuracy of the numerical integration as shown in Example 2.

Example 2 Evaluate the same hypersingular integral (39) as that used in Example 1 but take the singularity $y = 4.235 \times 10^{-3}$, which is close to the endpoint $x = 0$.

We first periodizate the integrand by the $\sin^p$-transformation [34]

$$\varphi_p(t) = \frac{\vartheta_p(t)}{\vartheta_p(1)}: [0, 1] \to [0, 1], \text{ where } \vartheta_p(t) = \int_0^t (\sin \pi s)^p \mathrm{d}s,\ p \in \mathbb{N}. \tag{40}$$

Table 3 Numerical results for the hypersingular integral $\int_0^1 \frac{(2x-1)^3}{(x-y)^2}\mathrm{d}x$ at $y = 4.235 \times 10^{-3}$

$n = \frac{1}{h}$	2^3	2^4	2^5	2^6	2^7
$e_n^{(0)}$	6.960e-0	5.149e-1	2.883e-2	1.776e-3	1.107e-4
$r_n^{(0)}$		$2^{3.756}$	$2^{4.158}$	$2^{4.020}$	$2^{4.000}$
$e_n^{(1)}$		8.527e-2	3.574e-3	2.721e-5	3.409e-7
$r_n^{(1)}$			$2^{4.576}$	$2^{7.037}$	$2^{6.318}$

Note that after using $\sin^p$-transformation to periodizate the integrand, the accuracy of numerical solutions is greatly improved comparing with non-periodic integral. The rates of error convergence match the error estimate (41)

Let $x = \varphi_3(t)$ in (39) and then apply quadratures (15) and (31) respectively to evaluate the transformed integral. Since $\varphi_3^{(i)}(t)|_{t=0,1} = 0$ for $1 \leq i \leq 3$, and $\varphi_3^{(i)}(t)|_{t=0,1} \neq 0$ for $i > 3$, the error analysis (16) gives

$$E_n(h) = c_2 h^4 + c_3 h^6 + ... + c_{m-1} h^{2m-2} + O(h^{2m}), \tag{41}$$

where $c_2, c_3, \ldots, c_{m-1}$ are constants independent of h. Numerical results are listed in Table 3, where $e_n^{(\mathrm{K})}$ are the absolute errors after the K-th Richardson extrapolation and $r_{2n}^{(\mathrm{K})} = e_n^{(\mathrm{K})}/e_{2n}^{(\mathrm{K})}$. We have

$$r_n^{(\mathrm{K})} = 2^{2\mathrm{K}+4},\ \mathrm{K} = 0, 1,$$

which coincide with the error estimate (41).

Remark 5 To verify the theoretical analysis, in this example, the singularity $y = 4.235 \times 10^{-3}$ is actually taken as the value of $\varphi_3(t)$ at $t = \frac{1}{8}$, which is one of the equally spaced abscissas. In practice, when y is arbitrary, one should first solve t from $\varphi_3(t) = y$ and then evaluate the integral efficiently by following the strategy in Remark 3.

Remark 6 Although the singularity $y = 4.235 \times 10^{-3}$ is much closer to the endpoint than $y = \frac{1}{64}$ in Example 1, the numerical results show that the accuracy of numerical solutions is greatly improved, which appreciates the periodization of the integral by utilizing $\sin^p$-transformation.

Example 3 Calculate the hypersingular integral with the fractional order singularity

$$(I(g))(y) = \int_0^1 \frac{g(x)}{\sqrt{|x-y|^3}} \mathrm{d}x,\ g(x) = (2x-1)^3. \tag{42}$$

The exact solution is

$$(I(g))(y) = -0.4 \left(\frac{128y^3 - 160y^2 + 60y - 5}{\sqrt{y}} + \frac{128y^3 - 224y^2 + 124y - 23}{\sqrt{1-y}} \right),$$

Table 4 Numerical results based on (21), (36) and (38) for $\int_0^1 \frac{(2x-1)^3}{\sqrt{|x-y|^3}} \mathrm{d}x$ at $y = \frac{1}{4}$

$n = \frac{1}{h}$	2^3	2^4	2^5	2^6	2^7	2^8	2^9
$e_n^{(0)}$	1.098e-1	3.889e-2	1.376e-2	4.867e-3	1.721e-3	6.087e-4	2.152e-4
$r_n^{(0)}$		$2^{1.497}$	$2^{1.499}$	$2^{1.499}$	$2^{1.499}$	$2^{1.499}$	$2^{1.499}$
$e_n^{(F)}$		8.784e-5	1.454e-5	3.127e-6	7.491e-7	1.852e-7	4.617e-8
$r_n^{(F)}$			$2^{2.594}$	$2^{2.217}$	$2^{2.061}$	$2^{2.015}$	$2^{2.004}$
$e_n^{(R)}$			9.885e-6	6.790e-7	4.358e-8	2.743e-9	1.717e-10
$r_n^{(R)}$				$2^{3.863}$	$2^{3.961}$	$2^{3.990}$	$2^{3.997}$

Note that the results verify the error estimate (43) where, $G(x) = \frac{g(x)}{\sqrt{|x-y|^3}}$ is a non-periodic function on [0, 1]. Since $g^{(i)}(x) = 0$ for $i \geq 4$, the error analysis (22) reads

$$E_n(h) = \sum_{\mu=1}^{m-1} c_\mu h^{2\mu} + d_1 h^{1.5} + O(h^{2m}), \tag{43}$$

where c_μ and d_1 are constants independent of h. The numerical results for (42) at $y = \frac{1}{4}$ and $y = \frac{1}{64}$ are listed in Tables 4–5. Where, $e_n^{(0)}$, $e_n^{(F)}$ and $e_n^{(R)}$ are, respectively, the absolute errors according to formulae (21), (36) and (38), and $r_{2n}^{(P)} = e_n^{(P)}/e_{2n}^{(P)}$ $(P = 0, F, R)$. It's obvious that

$$r_n^{(0)} \simeq 2^{1.5}, r_n^{(F)} \simeq 2^2, r_n^{(R)} \simeq 2^4,$$

which accords with the theoretical analysis (43).

Remark 7 As the value of y is close to the endpoint, the accuracy of numerical solutions decreases but the rate of convergence doesn't change. For example, it's seen than in Table 5, as quadrature (38) is applied, the error $e_n^{(0)} = 1.138 \times 10^{-5}$ as the number of abscissas $n = 2^9$ but the rate of convergence still matches the analysis (43). The accuracy can be improved by applying quadratures on the periodizated integral.

Here, we use $\sin^p$-transformation (40) to periodizate the integral by taking $x = \varphi_4(t)$ in (42). Since $\varphi_4^{(i)}(t)|_{t=0,1} = 0$ for $i = 1, ..., 4$, while $\varphi_4^{(i)}(t)|_{t=0,1} \neq 0$ for $i > 4$, the error estimate (22) gives

$$E_n(h) = c_1 h^{1.5} + c_2 h^{3.5} + O(h^4), \tag{44}$$

where c_1 and c_2 are constants independent of h. The numerical results of the hypersingular integral (42) at $y = 1.473 \times 10^{-3}$ are listed in Table 6, which shows

$$r_n^{(0)} \simeq 2^{1.5}, r_n^{(1)} \simeq 2^{3.5}.$$

The results verify the error analysis (44) exactly.

Table 5 Numerical results based on (21), (36) and (38) for $\int_0^1 \frac{(2x-1)^3}{\sqrt{|x-y|^3}} \mathrm{d}x$ at $y = \frac{1}{64}$

$n = \frac{1}{h}$	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
$e_n^{(0)}$	3.644e-2	10.34e-2	2.949e-3	8.608e-4	2.585e-4	7.990e-5	2.538e-5
$r_n^{(0)}$		$2^{1.818}$	$2^{1.809}$	$2^{1.777}$	$2^{1.736}$	$2^{1.694}$	$2^{1.655}$
$e_n^{(F)}$		3.942e-3	1.091e-3	2.814e-4	7.092e-5	1.777e-5	4.444e-6
$r_n^{(F)}$			$2^{1.853}$	$2^{1.956}$	$2^{1.988}$	$2^{1.997}$	$2^{1.999}$
$e_n^{(R)}$			1.413e-4	1.138e-5	7.750e-7	4.963e-8	3.121e-9
$r_n^{(R)}$				$2^{3.634}$	$2^{3.876}$	$2^{3.965}$	$2^{3.991}$

Note that the results verify the error estimate (43)

Table 6 Numerical results for $\int_0^1 \frac{(2x-1)^3}{\sqrt{|x-y|^3}}\mathrm{d}x$ at $y = 1.473 \times 10^{-3}$

$n = \frac{1}{h}$	2^3	2^4	2^5	2^6	2^7	2^8	2^9
$e_n^{(0)}$	6.033e-1	2.102e-1	7.421e-2	2.622e-2	9.271e-3	3.277e-3	1.158e-3
$r_n^{(0)}$		$2^{1.520}$	$2^{1.502}$	$2^{1.500}$	$2^{1.500}$	$2^{1.500}$	$2^{1.500}$
$e_n^{(1)}$		4.684e-3	2.021e-4	1.785e-5	1.565e-6	1.380e-7	1.219e-8
$r_n^{(1)}$			$2^{4.534}$	$2^{3.500}$	$2^{3.511}$	$2^{3.503}$	$2^{3.500}$
$e_n^{(2)}$			2.324e-4	1.457e-8	1.396e-8	3.595e-10	3.200e-12

After using $\sin^p$-transformation to periodizate the integrand, the accuracy of numerical solutions is greatly improved comparing with non-periodic integral and the rates of error convergence match the error estimates (44)

Remark 8 To verify the theoretical analysis, in this example, the singularity $y = 1.473 \times 10^{-3}$ is actually taken as the value of $\varphi_4(t)$ at $t = \frac{1}{8}$, which is one of the equally spaced abscissas. In practice, when y is arbitrary, one should first solve t from $\varphi_4(t) = y$ and then evaluate the integral efficiently by following Remark 3.

Remark 9 Although the singularity $y = 1.473 \times 10^{-3}$ is much closer to the endpoint than $y = \frac{1}{64}$, the numerical results show that the accuracy of numerical solutions is greatly improved, which appreciates the periodization of the integral by utilizing $\sin^p$-transformation.

Example 4 Calculate the periodic hypersingular integral

$$(I(g))(y) = \int_0^{2\pi} \frac{g(x)}{\sin^2((x-y)/2)}\mathrm{d}x, \;\; g(x) = \sin(2x), \tag{45}$$

where the exact solution is $(I(g))(y) = 2\sin(2y)$.

Note that $G(x) = \frac{\sin(2x)}{\sin^2((x-y)/2)}$ is a periodic function on $[0, 2\pi]$, the error estimate (27) reads

$$E_n(h) = O(h^{2m}) \tag{46}$$

Since $\sin(2x) \in C^\infty[0, 2\pi]$ with period $T = 2\pi$ and $G(x)$ is also infinitely differentiable on $\tilde{R} = (-\infty, = \infty)\backslash\{t + kT\}_{k=-\infty}^{+\infty}$with period $T = 2\pi$, the accuracy of numerical solutions is of exponential order.

Table 7 The numerical results for (45)

n	4	8	16	32	64	128
$e_{\max}$	1.11e-15	3.44e-15	2.38e-15	3.13e-15	2.78e-15	1.08e-15
$e_{\min}$	2.22e-16	1.11e-16	2.77e-16	2.56e-16	3.45e-16	1.87e-16

Note that since the integrand is periodic, the error converges exponentially

Table 8 The numerical results for $\int_{-1}^{1}\frac{\sqrt{1-x^2}}{(x-y)^2}\mathrm{d}x$ at y=0.125

$n=\frac{1}{h}$	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
$e_n^{(0)}$	8.431e-03	2.905e-03	1.012e-03	3.553e-04	1.251e-04	4.415e-05	1.560e-05
$r_n^{(0)}$		$2^{1.537}$	$2^{1.521}$	$2^{1.511}$	$2^{1.506}$	$2^{1.503}$	$2^{1.501}$
$e_n^{(1)}$		1.179e-04	2.253e-05	4.121e-06	7.402e-07	1.319e-07	2.340e-08
$r_n^{(1)}$			$2^{2.388}$	$2^{2.451}$	$2^{2.477}$	$2^{2.489}$	$2^{2.494}$
$e_n^{(2)}$			2.039e-06	1.683e-07	1.432e-08	1.240e-09	1.091e-10
$r_n^{(2)}$				$2^{3.599}$	$2^{3.555}$	$2^{3.529}$	$2^{3.506}$

Let $h=\frac{2\pi}{n}$ and $y_j=(2j-1)h$, the maximum error $e_{\max}=\max\limits_{1\le j\le n}|(I(g))(y_j)-(I_h(g))(y_j)|$ and the minimum error $e_{\min}=\min\limits_{1\le j\le n}|(I(g))(y_j)-(I_h(g))(y_j)|$, where $(I_h(g))(y_j)$ are the numerical solutions of the hypersingular integral (45) at $y=y_j$.

The numerical results listed in Table 7 shows that numerical results converge exponentially, which verifies the error analysis. Note that the accuracy cannot be improved by refining mesh due to the machine rounding error.

Example 5 In this example, we will not only test our method on a hypersingular integral, but also solve a hypersingular integral equation that often appears in the crack problems of the fracture mechanics. The good numerical results indicate that our quadrature is efficient and accurate, which matches the theoretical analysis; they also exhibit one of the main advantages of our method: the quadrature only depends on the unknown function itself but not its derivatives, so we can apply it straightforwardly for solving the boundary integral equations.

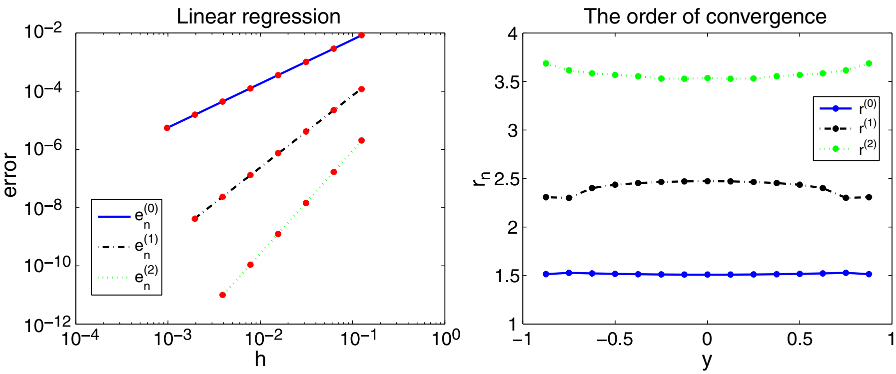

Fig. 1 (*Left*) Linear regression analysis at $y=0.125$. Note that the order of convergence matches error analysis and the order is clearly improved by using extrapolation. (*Right*) The order of convergence at different y evenly distributed on $(-1,1)$. Note that the order is stable and matches the error analysis

Table 9 The numerical results of D(0.125) for hypersingular BIE (48)

$n = \frac{1}{h}$	2^4	2^5	2^6	2^7	2^8	2^9
$e_n^{(0)}$	8.763e-04	2.389e-04	6.525e-05	1.777e-05	4.817e-06	1.299e-06
$r_n^{(0)}$		$2^{1.875}$	$2^{1.872}$	$2^{1.877}$	$2^{1.883}$	$2^{1.890}$

Note that second order convergence rate is obtained

Calculate the hypersingular integral

$$(I(g))(y) = \int_{-1}^{1} \frac{g(x)}{(x-y)^2}\mathrm{d}x,\ g(x) = \sqrt{1-x^2} \tag{47}$$

where, for any $y \in (-1, 1)$, the exact solution is $-\pi$ (see [5, 17]).

In Table 8, we list the numerical results at $y = 0.125$. Where the number of nodes $n = \frac{1}{h}$, $e_n^{(K)}$ (K = 0, 1, 2) are the absolute errors of the K-th Richardson extrapolation, and $r_{2n}^{(K)} = e_n^{(K)}/e_{2n}^{(K)}$.

The linear regression analysis shows $e_n^{(0)} = 0.191h^{1.51}$, $e_n^{(1)} = 0.021h^{2.47}$ after the fractional order extrapolation and $e_n^{(2)} = 0.003h^{3.53}$ after the second extrapolation. We show this linear regression analysis result graphically on Fig. 1. At the same time, to show the quadrature is not sensitive to the choice of y, the orders of convergence obtained from the linear regression analysis at different points y that evenly distribute on $(-1, 1)$ are also shown. The figure clearly states that the method is stable.

Next, we consider solving the following hypersingular integral equation by the quadrature presented in this paper:

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{(x-y)^2} D(x)\mathrm{d}x = -\pi,\ -1 < y < 1, \tag{48}$$

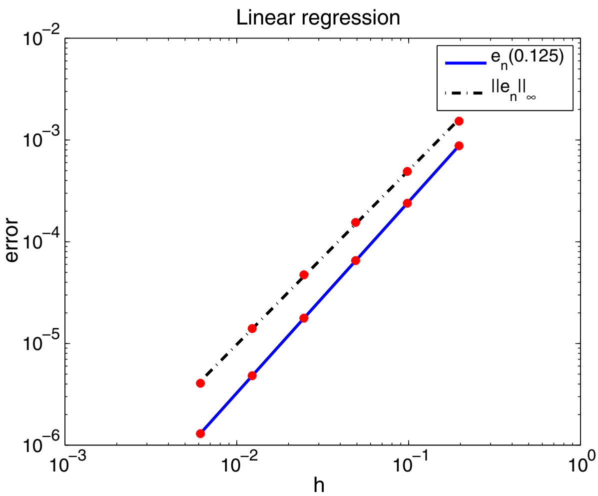

Fig. 2 Linear regression analysis for error of $D(x)$ at $x = 0.125$, $e_n(0.125)$ and maximum error of $D(x)$, $\|e_n(x)\|_\infty$. Note that the result is accurate and is converged at the rate of $O(h^2)$

where $D(x)$, the density function, is unknown and its exact value should be 1 on $(-1, 1)$. Solving such type of integral equations is often needed in the application of the boundary integral equation method to crack problems in fracture mechanics (see [5, 17, 24]). Here, we use (21) to discretize the left hand side. The numerical errors of $D(x)$ at $x = 0.125$ is listed in the Table 9. Figure 2 shows the linear regression analysis for both $e_n(0.125)$, error of $D(0.125)$; and $\|e_n(x)\|_\infty$, the maximum error of $D(x)$, when the number of nodes in quadrature is n. The numerical results clearly show the nearly second order convergence is obtained by using (21). More detailed analysis in the context of hypersingular boundary integral equation will be our next future work.

5 Conclusions

From the presented theoretical and numerical results, we can draw the following conclusions.

1. For calculating hypersingular integrals, the algorithms are straightforward, without need to calculate any weight;
2. The numerical experiments verify the theoretical error analysis, which shows the accuracy order of the algorithms is high, especially, for hypersingular periodic functions;
3. The accuracy of the algorithms can be greatly improved by the Richardson extrapolation;
4. Example 5 shows the direct application of the quadratures for solving corresponding hypersingular boundary integral equations. Since the quadratures only involve the unknown density function, but not its derivatives, the quadrature formulae presented in this paper can be easily applied. The numerical results also show the effectiveness of the quadrature methods. Corresponding analyses and applications will be presented in a separate paper.

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