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EXTRAPOLATION ALGORITHMS FOR SOLVING MIXED BOUNDARY INTEGRAL EQUATIONS OF THE HELMHOLTZ EQUATION BY MECHANICAL QUADRATURE METHODS*

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Abstract. This paper presents mechanical quadrature methods with high accuracy for solving mixed boundary integral equations of the Helmholtz equation. By estimating the range of eigenvalues for the discretization matrix of the integral equations and applying the collectively compact convergent theory, we prove the stability and convergence of numerical solutions, which is a challenging task for this method. Moreover, the asymptotic error expansions show the method is of order h^3. Hence, extrapolation algorithms can be introduced to achieve higher approximation accuracy degree ($\mathcal{O}(h^5)$). Meanwhile, an a posteriori asymptotic error estimate is derived, which can be used to construct self-adaptive algorithms. The numerical examples support our theoretical analysis.

Key words. Helmholtz equation, mechanical quadrature method, extrapolation algorithm, a posteriori estimate

AMS subject classifications. 35J05, 65N38, 65R20

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1. Introduction.

Time-harmonic acoustic wave scattering or radiation by a cylindrical obstacle is essentially a two-dimensional problem and is often described in acoustic media by the Helmholtz equation with associated boundary conditions. Depending on the sound-absorbing behavior of the scattering obstacle, Neumann, Robin, or Dirichlet conditions may be imposed. In this paper we consider the following mixed boundary value problems:

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 \quad \text{in} \quad \Omega', \\ \alpha u(x) + \beta \frac{\partial u(x)}{\partial n} = g(x) \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1a}$$

where $\Omega \subset \Re^2$ is a bounded domain with the smooth boundary $\partial\Omega$, and $\Omega' = \Re^2 \backslash \Omega$. In (1.1a), α and β are known constants; g is a known function on $\partial\Omega$; k is the acoustic wave number; and n is the outward unit normal to $\partial\Omega$. If $x \in \Omega'$, (1.1a) satisfies the following Sommerfeld radiation condition:

$$\frac{\partial u(x)}{\partial r} - iku(x) = o(r^{-1/2}), \quad r = |x| \to \infty. \tag{1.1b}$$

By Green's formula, (1.1) can be converted into the following boundary integral equation (BIE) [6]:

$$-\int_{\partial\Omega} \frac{\partial u(x)}{\partial n_x} \Phi(x,y) ds_x + \int_{\partial\Omega} u(x) \frac{\partial}{\partial n_x} \Phi(x,y) ds_x = \theta(y) u(y). \tag{1.2a}$$

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Here, in particular, $\theta(y) = \frac{1}{2\pi}$ as long as the boundary $\partial\Omega$ is smooth; $|x-y| = [(x_1-y_1)^2+(x_2-y_2)^2]^{1/2}$; and $\Phi(x,y)$, the fundamental solution of the Helmholtz equation, is given by

$$\Phi(x,y) = \frac{i}{4}H_0^{(1)}(k|x-y|), \quad x \neq y. \tag{1.2b}$$

$H_0^{(1)} = J_0 + iN_0$ is the Hankel function of order zero and of the first kind, where

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\Big(\frac{z}{2}\Big)^{2n} \tag{1.2c}$$

for the Bessel function of order zero and

$$N_0(z) = \frac{2}{\pi}\Big(\ln\frac{z}{2}+c\Big)J_0(z) + \frac{2}{\pi}\sum_{n=1}^{\infty}\Big(\sum_{m=1}^{n}\frac{1}{m}\Big)\frac{(-1)^{n+1}}{(n!)^2}\Big(\frac{z}{2}\Big)^{2n} \tag{1.2d}$$

for the Neumann function of order zero. $c = 0.57721\ldots$ represents Euler's constant.

Then, to analyze properties of the kernel in the following context, we can decompose the kernel by

$$\Phi(x,y) = k_1(x,y) + k_2(x,y), \tag{1.3}$$

where $k_1(x,y) = -\frac{1}{2\pi}\ln|x-y|$ is logarithmically singular and $k_2(x,y) = \frac{i}{4} - \frac{1}{2\pi}(\ln\frac{k}{2} + c) + O(|x-y|\ln|x-y|)$ is smooth. If one of $u(x)$ and $\frac{\partial u(x)}{\partial n_x}$ is given by the boundary conditions of (1.1), then the other can be solved by (1.2a). Once both $u(x)$ and $\frac{\partial u(x)}{\partial n_x}$ $(x \in \partial\Omega)$ are known, the solution $u(y)$ $(y \in \Omega')$ can be calculated by

$$u(y) = \frac{1}{2\pi}\int_{\partial\Omega} u(x)\frac{\partial}{\partial n_x}\Phi(x,y)ds_x - \frac{1}{2\pi}\int_{\partial\Omega}\frac{\partial u(x)}{\partial n_x}\Phi(x,y)ds_x. \tag{1.4}$$

It is known that if $\beta = 0$ on $\partial\Omega$, then (1.1a) becomes a pure Dirichlet problem, and (1.2a) is the weakly singular BIE system of the first kind, whose solution exists and is unique as long as $C_\Gamma \neq 1$, where C_Γ is the logarithmic capacity (i.e., the transfinite diameter); if $\alpha = 0$ on $\partial\Omega$, then (1.1a) becomes a pure Neumann problem, and (1.2a) is the weakly singular BIE system of the second kind, whose solution exists if and only if

$$\int_{\partial\Omega} g(x)/\beta ds_x = 0; \tag{1.5}$$

if $\alpha\beta > 0$ on $\partial\Omega$, and both the Dirichlet and the Neumann boundary conditions are assigned on $\partial\Omega$, then (1.2a) is a weakly singular mixed BIE system, which has a unique solution.

A considerable part of the research on the numerical solution of (1.2a) is concerned with applications of Galerkin methods and collocation methods [2, 3, 4, 13, 16, 17] and qualocation methods [20] (using splines and trigonometric polynomials as trial functions). Yan [22] established a fast boundary element method for Dirichlet conditions on (1.2a). Shen and Wang [18] gave spectral approximations of the Helmholtz equations with high wave numbers. Kress and Sloan [14] gave a quadrature method for solving the BIE of the first kind on (1.2a), and their error analysis was based

on trigonometric interpolation theory. Kress [13] also discussed the quadrature or Nyström method for the approximate solution of integral equations of the second kind with continuous or weakly singular kernels. He pointed out that, in general, if a suitable numerical quadrature scheme is available for the Nyström method, then it will be superior in efficiency to the collocation method, which requires integration for its matrix elements, and to the Galerkin method, which requires double integration.

The mechanical quadrature method (MQM), for solving the BIE, is constructed by using a new type of numerical quadrature but not the Nyström interpolation. It not only preserves the advantage of requiring less computational cost than projection methods since each element in the discretization matrix of integral equations is evaluated directly by the mechanical quadrature, but it also can be applied for solving the first-kind BIE. Some related work can be found in [10, 12, 23].

The extrapolation algorithm (EA) based on asymptotic expansion of errors is an effective parallel algorithm because it possesses a high degree of accuracy, good stability, and almost optimal computational complexity. It has been applied to many problems, such as numerical integration, finite difference methods, and finite element methods [15]. Using Galerkin methods, Rüde and Zhou [17] established multiparameter extrapolation methods for the BIE system of the second kind on polygonal domains. Assuming that Ω is a bounded and simply connected open region with a smooth boundary $\partial\Omega$, and that the inverse matrix of discrete equations exists and is uniformly bounded, Xu and Zhao [21] established an extrapolation method to solve the BIE for the boundary value problem of the third kind. We have established splitting extrapolations for solving the BIE of the first kind on the Laplace equation [11] and of elasticity [12] on polygons by the MQM [10]. Also, we presented MQMs and their splitting extrapolations to solve BIEs of axisymmetric Laplace mixed boundary value problems [23].

In this paper, we construct MQMs and corresponding EAs for (1.2a). First, we use the quadrature rules presented by Sidi and Israeli [19] to calculate weakly singular integrals. Second, using spectral analysis, we prove that the solution of the discrete equation exists and is unique. Then we show that the condition number is $O(h^{-1})$. Third, using perturbation theory and Anselon's collective compact theory [1], we prove the $O(h^3)$ convergence rate. Finally, using the asymptotic expansion of errors, we establish the EA. The first extrapolation step improves the convergence rate to $O(h^5)$. We also derive an a posteriori error estimate for the self-adaptive algorithm.

This paper is organized as follows: In section 2, we present the MQM and the convergence theory. In section 3, we construct the EA, and we provide the asymptotic expansion of errors and an a posteriori error estimate. Numerical examples are performed in section 4, and some conclusions are listed in section 5.

2. MQMs for solving BIEs. Define boundary integral operators on $\partial\Omega$,

$$(Aw)(y) = \int_{\partial\Omega} w(x)\Phi(x,y)ds_x, \quad y \in \partial\Omega, \tag{2.1a}$$

and

$$(Kw)(y) = \int_{\partial\Omega} w(x)\frac{\partial}{\partial n_x}\Phi(x,y)ds_x, \quad y \in \partial\Omega, \tag{2.1b}$$

where

$$w(x) = \begin{cases} \frac{\partial u(x)}{\partial n}, \ x \in \partial\Omega, \ \text{if } \beta(x) = 0 \text{ on } \partial\Omega, \\ u(x), \ x \in \partial\Omega, \ \text{if } \beta(x) \neq 0 \text{ on } \partial\Omega. \end{cases}$$

Then (1.2a) can be converted into a matrix operator equation

$$\beta\theta(y)w + \alpha Aw + \beta Kw = f, \tag{2.2}$$

where $\theta(y) = 1/(2\pi)$, and

$$f(y) = \begin{cases} \theta(y)g(y) \text{ if } \beta = 0, \\ (Ag)(y) \text{ if } \beta \neq 0. \end{cases}$$

Assume the parameter mapping $x = x(s) = (x_1(s), x_2(s)) \in C^{2l+1}[0, 2\pi)$ maps $[0, 2\pi)$ onto $\partial\Omega$ with $|x^{'}(s)|^2 = |x_1^{'}(s)|^2 + |x_2^{'}(s)|^2 > 0$. With the same assumptions on the parameter mapping, $y = x(t) = (x_1(t), x_2(t))$, (2.2) can be expressed by

$$(\beta\theta(y)E + \alpha A_0 + \alpha A_1 + \alpha A_2 + \beta B)v = F, \tag{2.3}$$

where E is the identity operator,

$$(A_0 v)(t) = \int_0^{2\pi} a_0(t, s)v(s)ds, \quad t \in [0, 2\pi), \tag{2.4a}$$

with $a_0(t, s) = -\frac{1}{2\pi} \ln |2e^{-1/2} \sin \frac{(t-s)}{2}|$,

$$(A_1 v)(t) = \int_0^{2\pi} a_1(t, s)v(s)ds, \quad t \in [0, 2\pi), \tag{2.4b}$$

with $a_1(t, s) = -\frac{1}{2\pi}[\ln |x(t) - x(s)| - \ln |2e^{-1/2} \sin \frac{(t-s)}{2}|]$,

$$(A_2 v)(t) = \int_0^{2\pi} a_2(t, s)v(s)ds, \quad t \in [0, 2\pi), \tag{2.4c}$$

with $a_2(t, s) = k_2(x(t), x(s))$, and

$$(Bv)(t) = \int_0^{2\pi} b(t, s)v(s)ds, \quad t \in [0, 2\pi), \tag{2.4d}$$

with $b(t, s) = \frac{\partial}{\partial n_x}\Phi(x(t), x(s))$, $v(t) = w(x(t))|x'(t)|$, and $|x(t) - x(s)|^2 = (x_1(t) - x_1(s))^2 + (x_2(t) - x_2(s))^2$:

$$F(t) = f(x(t)) = \begin{cases} g(t) \text{ if } \beta = 0, \\ ((A_0 + A_1 + A_2)g)(t) \text{ if } \beta \neq 0, \end{cases}$$

and $g(t) = g(x(t))|x'(t)|$. Obviously, when $\partial\Omega$ is smooth, the integral kernels $a_1(t, s)$, $a_2(t, s)$, and $b(t, s)$ are all sufficiently smooth functions.

Let $h = 2\pi/n$ ($n \in N$) be the mesh width and $t_j = jh$ ($j = 0, 1, \ldots, n$) be nodes. For an integral operator D with a periodic smooth kernel $d(t, s)$, we construct the following Nyström approximation by the midpoint or the trapezoidal rule [8]:

$$(D^h v)(t) = h \sum_{j=0}^{n} d(t, s_j)v(s_j), \quad t \in [0, 2\pi), \tag{2.5a}$$

which has the following error bounds:

$$(Dv)(t) - (D^h v)(t) = O(h^{2l}), \quad l \in N. \tag{2.5b}$$

For the weakly singular operators A_0, by the quadrature formula [19], we construct the Fredholm approximation,

$$(A_0^h v)(t_i) = -\frac{h}{2\pi}\left[\sum_{j=0, i\neq j}^{n} \ln\left|2e^{-1/2}\sin\frac{(i-j)h}{2}\right| v(s_j)\right] - \frac{h}{2\pi}\ln\left|2\pi e^{-1/2}\frac{h}{2\pi}\right| v(t_i), \quad i = 0, 1, \ldots, n, \tag{2.6a}$$

which has the following error bounds [19]:

$$(A_0^h v)(t_i) - (A_0 v)(t_i) = \frac{-2}{\pi}\sum_{\mu=1}^{2l-1}\frac{\zeta'(-2\mu)}{(2\mu)!}[v(t)]^{(2\mu)}|_{t=t_i} h^{2\mu+1} + O(h^{2l}), \tag{2.6b}$$

where $\zeta'(t)$ is the derivative of the Riemann Zeta function.

Consider the discrete approximation of (2.3)

$$(\beta\theta(y)E^h + \alpha A_0^h + \alpha A_1^h + \alpha A_2^h + \beta B^h)v^h = F^h, \tag{2.7a}$$

where $v^h = (v_0^h, \ldots, v_n^h)^T$, E^h is the identity matrix, $A_m^h = [a_m(t_i, s_j)]_{i,j=0}^n$ $(m = 0, 1, 2)$, $B^h = [b(t_i, s_j)]_{i,j=0}^n$, and $F^h = (f(x(t_0)), \ldots, f(x(t_n)))^T$ with

$$f(x(t_i)) = \begin{cases} g(t_i) \text{ if } \beta = 0, \\ ((A_0^h + A_1^h + A_2^h)g)(t_i) \text{ if } \beta \neq 0. \end{cases} \tag{2.7b}$$

Obviously, (2.7) is a linear equation system with n unknowns. Once v^h is determined from (2.7), the solution $u(y)$ $(y \in \Omega')$ can be computed by

$$u(y) = \frac{h}{2\pi}\sum_{j=0}^{n}\left[v^h(t_j)\frac{\partial}{\partial n_x}\Phi(x(t_j), y) - v^h(t_j)\Phi(x(t_j), y)\right]|x'(t_j)|. \tag{2.8}$$

To show that the solution of (2.7) is unique, we first prove that the operator $\beta\theta(y)E^h + \alpha A_0^h$ is invertible (Corollary 2.2) by estimating the upper and lower bounds of eigenvalues of the discretization matrix of the integral equations (Lemma 2.1). Then, in Lemma 2.3, we show the uniform boundedness and convergence of the product of the inverse discretization matrix A_0^h and the operator A_0. After that, through the proof that Nyström approximation is a collectively compact operator (Lemma 2.4 and Corollary 2.5), we prove the existence and convergence of numerical approximations (Theorem 2.6 and Corollary 2.7).

From (2.6), we have $A_0^h = \{-\frac{1}{2\pi}h|\ln 2\pi e^{-1/2}\frac{h}{2\pi}|, -\frac{1}{2\pi}h\ln|2e^{-1/2}\sin\frac{h}{2}|, \ldots, -\frac{1}{2\pi}h\ln|2e^{-1/2}\sin\frac{nh}{2}|\}$.

LEMMA 2.1. *The eigenvalues λ_k of A_0^h are positive, and*

$$1/(4\pi n) \leq \lambda_k \leq c \ (k = 0, \ldots, n-1), \tag{2.9}$$

where c is a constant independent of h.

Proof. Since A_0^h is a symmetric circulant matrix [8], we have $\lambda_k = F(\varepsilon^k)/(2\pi)$ with $\varepsilon^k = \exp(2\pi ki/n)$ and

$$F(z) = -h\left[\ln|he^{-1/2}| + \sum_{j=1}^{n-1} z^j \ln|2e^{-1/2}\sin(j\pi/n)|\right]. \tag{2.10}$$

For $k = 0$, we first prove the following identity:

$$\Pi_{j=1}^{n-1} \sin\left(\frac{j\pi}{n}\right) = \frac{n}{2^{n-1}} \quad (n \geq 2). \tag{2.11}$$

Since the identity

$$x^n - 1 = (x-1)\Pi_{j=1}^{n-1}(x - e^{2\pi ij/n}),$$

$$1 + x + x^2 + \cdots + x^{n-1} = \Pi_{j=1}^{n-1}(x - e^{2\pi ij/n}),$$

letting $x = 1$, we complete the proof of (2.11). Using (2.11), we derive

$$F(1) = -h\left[\ln|he^{-1/2}| + \sum_{j=1}^{n-1} \ln|2e^{-1/2}\sin(j\pi/n)|\right] = \pi - h\ln(2\pi) < \pi,$$

and $\lambda_0 = \frac{\pi - h\ln(2\pi)}{2\pi} < \frac{1}{2}$. For $k = 1, \ldots, n-1$, we have

$$\begin{aligned} \lambda_k &= -h[\ln|he^{-1/2}| + \Psi(\varepsilon^k)]/(2\pi) \\ &= (\ln(2n) + (1/2 - \ln 2)\Phi(\varepsilon^k) - \Psi(\varepsilon^k))/(2\pi n), \end{aligned} \tag{2.12}$$

where $\Phi(z) = 1 + z + z^2 + \cdots + z^{n-1}$, and $\Psi(z) = \sum_{j=1}^{n-1} z^j \ln|\sin(j\pi/n)|$. Note that since $\Phi(\varepsilon^k) = n\delta_{k,0}$, and

$$|\Psi(z)| \leq \sum_{j=1}^{n-1} -\ln|\sin(j\pi/n)| = -\ln(\Pi_{j=1}^{n-1}\sin(j\pi/n)) = -\ln n + (n-1)\ln 2$$

with $n \geq 3$, we get the upper bound for $k = 1, \ldots, n-1$,

$$\lambda_k \leq \frac{1}{2\pi n}(\ln(2n) - \ln n + (n-1)\ln 2) = \frac{1}{2\pi}\ln 2. \tag{2.13}$$

To estimate the lower bound of λ_k, consider

$$\begin{aligned} &\ln|e^{-1/2}/n| + \sum_{j=1}^{n-1} \cos(2kj\pi/n)\ln|2e^{-1/2}\sin(j\pi/n)| \\ &\qquad = -\ln n + \sum_{j=1}^{n-1} \cos(2kj\pi/n)\ln|2\sin(j\pi/n)|. \end{aligned}$$

Using the expansions of the ψ-special function [7]

$$\psi(k/n) = -\gamma - \ln n - \pi/2\cot(k\pi/n) + \sum_{j=1}^{n} \cos(2kj\pi/n)\ln|2\sin(j\pi/n)|$$

and $\psi(z) = -\gamma - 1/z + z\sum_{j=1}^{\infty} 1/[j(j+z)]$, we obtain

$$\sum_{j=1}^{n-1} \cos(2kj\pi/n)\ln|2\sin(j\pi/n)| = \ln n + \pi/2\cot(k\pi/n) - n/k + k/n\sum_{j=1}^{\infty}[j(j+k/n)]^{-1}$$

and

(2.14)
$$\lambda_k = -\frac{1}{2\pi n}\left\{\pi/2\cot(k\pi/n) - n/k + k/n\sum_{j=1}^{\infty}[j(j+k/n)]^{-1}\right\} \quad (1 \le k \le n-1),$$

where γ is Euler's constant. Substituting

$$\cot(k\pi/n) = n/(k\pi) - k\pi/(3n) - 1/45(k\pi/n)^3 - \cdots - 2^{2j}B_j/(2j)!(k\pi/n)^{2j-1} - \cdots$$

into (2.14), we get

$$\lambda_k = -\frac{1}{2\pi n}\Big\{ - n/(2k) - k\pi^2/(6n) - \cdots - 2^{2j-1}B_j/(2j)!(k\pi/n)^{2j-1}\pi \\ - \cdots + k/n\sum_{j=1}^{\infty}[j(j+k/n)]^{-1}\Big\}$$

and

$$\lambda_k = \Big\{1/(2k) + k\pi^2/(6n^2) + \cdots + 2^{2j-1}B_j/(2j)!(k\pi/n)^{2j-1}\pi/n \\ + \cdots - k/n^2\sum_{j=1}^{\infty}[j(j+k/n)]^{-1}\Big\}/(2\pi),$$

where B_j is the jth Bernoulli number. Using

$$k\pi^2/(6n^2) - k/n^2\sum_{j=1}^{\infty}[j(j+k/n)]^{-1} > k/n^2\left\{\sum_{j=1}^{\infty}[j^{-2} - (j(j+k/n))^{-1}]\right\} > 0,$$

we obtain for $k = 1, 2, \ldots, n-1$

$$\lambda_k > 1/(4\pi k) + (k/n)^3/(180n) + \cdots > 1/(4\pi k) > 1/(4\pi n).$$

Combining the upper and lower bounds, we complete the proof of Lemma 2.1. ∎

COROLLARY 2.2. (1) *If $\beta = 0$, then αA_0^h is invertible and*

$$||(\alpha A_0^h)^{-1}|| = O(n). \tag{2.15}$$

(2) *If $\beta \neq 0$ and $\beta\alpha \ge 0$, then $\beta\theta(y)E^h + \alpha A_0^h$ is invertible and $(\beta\theta(y)E^h + \alpha A_0^h)^{-1}$ is uniformly bounded, i.e.,*

$$||(\beta\theta(y)E^h + \alpha A_0^h)^{-1}|| \le |(\beta\theta(y))^{-1}|, \tag{2.16}$$

where $\|\cdot\|$ denotes the spectral norm.

Proof. (1) If $\beta = 0$, using Ostrowski's theorem [9], we have $\lambda_k(\alpha A_0^h) \ge \min_{x\in\partial\Omega}|\alpha|$, $\min_{0\le k\le n}\lambda_k(A_0^h)$, and $||(\alpha A_0^h)^{-1}|| = O(n)$.

(2) If $\beta \neq 0$ and $\beta\alpha \ge 0$, from Lemma 2.1, we obtain $|\lambda_k(\beta\theta(y)E^h + \alpha A_0^h)| = |\beta\theta(y) + \alpha\lambda(A_0^h)| \ge |\beta\theta(y)|$. Then (2.16) holds. ∎

Using Corollary 2.2, (2.7) is equivalent to

$$(E^h + (\beta\theta(y)E^h + \alpha A_0^h)^{-1}(\alpha A_1^h + \alpha A_2^h + \beta B^h))v^h = (\beta\theta(y)E^h + \alpha A_0^h)^{-1}F^h. \tag{2.17}$$

To discuss the existence and convergence of numerical approximations, we first introduce some special operators. Let $S^h = \text{span}\{e_j(t), j = 0, 1, \ldots, n\} \subset C[0, 2\pi)$ be a piecewise linear function subspace with nodes $\{t_i\}_{i=0}^n$, where $e_j(t)$ is the basis function satisfying $e_j(t_i) = \delta_{ji}$. Define a prolongation operator $I^h : \Re^n \to S^h$ satisfying

$$I^h v = \sum_{j=0}^{n} v_j e_j(t) \quad \forall v = (v_0, \ldots, v_n) \in \Re^n \tag{2.18}$$

and a restricted operator $R^h : C[0, 2\pi) \to \Re^n$ satisfying

$$R^h \rho = (\rho(t_0), \ldots, \rho(t_n)) \in \Re^n \quad \forall \rho \in C[0, 2\pi). \tag{2.19}$$

For the convergence, we have the following lemma.

LEMMA 2.3. *The operator sequence* $\{I^h (A_0^h)^{-1} R^h A_0 : C^3[0, 2\pi) \to C[0, 2\pi)\}$ *is uniformly bounded and convergent to the embedding operator* I.

Proof. From the quadrature rule (2.6), let $\phi \in C^3[0, 2\pi)$ and ϕ^h be solutions of the auxiliary equations $A_0\phi = \rho$ and $A_0^h \phi^h = R^h \rho$, respectively. We have

$$A_0\phi(t_i) = A_0^h \phi(t_i) + \varepsilon_i,$$

where $\varepsilon_i = O(h^3)$ $(i = 0, \ldots, n)$. Letting $e(t_i) = \phi^h(t_i) - \phi(t_i)$, we obtain

$$\begin{aligned} A_0^h e(t_i) &= A_0^h (\phi^h - \phi)|_{t=t_i} \\ &= R^h \rho|_{t=t_i} - (A_0\phi(t_i) - \varepsilon_i) \\ &= \varepsilon_i = O(h^3). \end{aligned}$$

That is,

$$A_0^h e = \varepsilon, \quad e^T = (e(t_0), \ldots, e(t_n)), \quad \varepsilon^T = (\varepsilon_0, \ldots, \varepsilon_n). \tag{2.20}$$

Thus from Lemma 2.1 and Corollary 2.2, we have $e = (A_0^h)^{-1}\varepsilon$ with $||e|| = O(h^2)$, and

$$\begin{aligned} ||e|| &= ||(A_0^h)^{-1}\varepsilon|| \\ &= ||R^h A_0^{-1}\rho - (A_0^h)^{-1} R^h \rho|| \\ &= ||R^h \phi - (A_0^h)^{-1} R^h A_0 \phi||. \end{aligned} \tag{2.21}$$

By $I^h R^h \to I$ as $h \to 0$ in $\mathfrak{L}(C^3[0, 2\pi) \to C[0, 2\pi))$, the proof of Lemma 2.3 is completed. ∎

LEMMA 2.4. *Let the integral operator* $A \in \mathfrak{L}(C[0, 2\pi), C^k[0, 2\pi))$ *and* A^{-1} *exist, and let the integral operator* $D \in \mathfrak{L}(C[0, 2\pi),\ C^{k+1}[0,\ 2\pi))$ *with the kernel* $d(t, \tau)$ *of* D *satisfy* $A^{-1}d(\cdot, \tau) = \tilde{d}(t, \tau)$. *Also assume that* $\tilde{d}(t, \tau)$ *and* $\frac{d}{dt}\tilde{d}(t, \tau)$ *are continuous on* $[0, 2\pi)^2$. *Then the Nyström approximation of* $\tilde{D}$,

$$\tilde{D}^h u = h \sum_{j=1}^{n} \tilde{d}(t, \tau_j) u(\tau_j) \quad \forall u \in C[0, 2\pi), \tag{2.22}$$

is the collectively compact convergent to $\tilde{D}$, *i.e.,*

$$A^{-1} D^h \overset{c.c}{\to} A^{-1} D \quad \textit{in} \quad \mathfrak{L}(C^0[0, 2\pi), C^1[0, 2\pi)). \tag{2.23}$$

Proof. From (2.22), we have

$$A^{-1}D^h u = h\sum_{j=1}^{n}\tilde{d}(t,\tau_j)u(\tau_j) \to \int_0^{2\pi}\tilde{d}(t,\tau)u(\tau)d\tau = A^{-1}Du.$$

Since $d(t,\tau)$ is continuous on $[0,2\pi)^2$, we have [1, 5]

$$A^{-1}D^h \overset{c.c}{\to} A^{-1}D \quad \text{in} \quad \mathfrak{L}(C^0[0,2\pi), C^\infty[0,2\pi))$$

and

$$\frac{d}{dt}A^{-1}D^h u = \frac{d}{dt}\left[h\sum_{j=1}^{n}\tilde{d}(t,\tau_j)u(\tau_j)\right].$$

Based on the continuity of $\frac{d}{dt}\tilde{d}(t,\tau)$, (2.23) holds. ∎

COROLLARY 2.5. *Let the Nyström approximation D^h be defined by* (2.5) *and suppose $C_\Gamma \neq 1$.*

(1) *For $\beta = 0$, we have*

$$I^h(\alpha A_0^h)^{-1}R^h D^h \overset{c.c}{\to} (\alpha A_0)^{-1}D \quad in \quad C[0,2\pi) \to C[0,2\pi), \tag{2.24}$$

and $E^h + I^h(\alpha A_0^h)^{-1}R^h D^h$ is invertible, and its inverse operator is uniformly bounded.

(2) *For $\beta \neq 0$, we have*

$$I^h(\beta\theta(y)E^h + \alpha A_0^h)^{-1}R^h D^h \overset{c.c}{\to} (\beta\theta(y)E + \alpha A_0)^{-1}D \quad in \quad C[0,2\pi) \to C[0,2\pi), \tag{2.25}$$

and $E^h + I^h(\beta\theta(y)E^h + \alpha A_0^h)^{-1}R^h D^h$ is invertible, and its inverse operator is uniformly bounded.

Proof. Since for $\beta = 0$ and $C_\Gamma \neq 1$ the kernel $d(t,\tau)$ of the operator D and its derivatives of higher order are continuous [2, 3, 4], and we have

$$||I^h(\alpha A_0^h)^{-1}R^h D^h||_{0,0} \leq ||I^h(\alpha A_0^h)^{-1}R^h(\alpha A_0)||_{0,3}||(\alpha A_0)^{-1}D^h||_{3,0},$$

by Lemma 2.4 and $(\alpha A_0)^{-1}D^h \in \mathfrak{L}(C[0,2\pi), C^3[0,2\pi))$, there exists a constant c such that

$$||(\alpha A_0)^{-1}D^h||_{3,0} \leq c \quad \text{and} \quad ||I^h(\alpha A_0^h)^{-1}R^h(\alpha A_0)||_{0,3} \leq c,$$

where $||\cdot||_{m_2,m_1}$ is the norm of the linear bounded operator space $\mathfrak{L}(C^{m_1}[0,2\pi), C^{m_2}[0,2\pi))$. Using the results of [1, 5, 13], the operator sequence $\{(\alpha A_0)^{-1}D^h : C[0,2\pi) \to C^3[0,2\pi)\}$ must be collectively compactly convergent to $(\alpha A_0)^{-1}D$. Hence, we complete the proof of (1). Similarly, we can prove (2). ∎

Replacing $(M^h)^{-1} = (\beta\theta(y)E^h + \alpha A_0^h)^{-1}$, αA_1^h, αA_2^h, and βB^h by $(\hat{M}^h)^{-1} = I^h(M^h)^{-1}R^h$, $\hat{A}_1^h = I^h(\alpha A_1^h)R^h$, $\hat{A}_2^h = I^h(\alpha A_2^h)R^h$, and $\hat{B}^h = I^h(\beta B^h)R^h$, respectively, we obtain the operator

$$\hat{L}^h = I^h(M^h)^{-1}R^h(\alpha A_1^h + \alpha A_2^h + \beta B^h)R^h.$$

Now consider the operator equation

$$(E^h + \hat{L}^h)\hat{v}^h = \hat{F}^h \tag{2.26}$$

with $\hat{F}^h = I^h(\beta\theta(y)E^h + \alpha A_0^h)^{-1}R^hF^h$. Obviously, if $\hat{v}^h = I^hv^h$ is a solution of (2.26), then $R^h\hat{v}^h$ must be a solution of (2.17); conversely, if v^h is a solution of (2.17), then I^hv^h must be a solution of (2.26). Below we prove that there exists a unique solution $\hat{v}^h$ in (2.26) such that $\hat{v}^h$ converges to v.

THEOREM 2.6. *The operator sequence $\{\hat{L}^h\}$ is collectively compactly convergent to $L = (\beta\theta(y)E + \alpha A_0)^{-1}(\alpha A_1 + \alpha A_2 + \beta B)$ in $C[0, 2\pi)$, i.e.,*

$$\hat{L}^h \overset{c.c}{\to} L. \tag{2.27}$$

Proof. We first prove that $\{\hat{L}^h\}$ is a collectively compact operator sequence in $C[0, 2\pi)$. Let $\{Z_h\}_{h\in H}$ be the bounded sequence, where $Z_h = \{z_h\}$ and $H = \{h_1, h_2, \ldots\}$ and $h_j \to 0$ as $j \to \infty$. Below we verify that there exists a convergent subsequence in $\{\hat{L}^hZ_h\}$. So we consider the convergence of

$$\{I^h(M^h)^{-1}R^h(\alpha A_1^h + \alpha A_2^h + \beta B^h)R^hz_h\}. \tag{2.28}$$

Using the results of Lemma 2.4 and Corollary 2.5, and by

$$I^h(M^h)^{-1}R^h\hat{A}_1^hz_h = I^h(M^h)^{-1}R^h(\beta\theta(y)E^h + \alpha A_0^h)[(\beta\theta(y)E^h + \alpha A_0^h)^{-1}R^h(\alpha A_1^h)]z_h,$$

we have

$$\begin{aligned}
&||I^h(M^h)^{-1}R^hD^h||_{0,0}\\
&\le ||I^h(M^h)^{-1}R^h(\beta\theta(y)E^h + \alpha A_0^h)||_{0,1}\cdot||(\beta\theta(y)E^h + \alpha A_0^h)^{-1}R^hD^h||_{1,0}\\
&\le c||(\beta\theta(y)E^h + \alpha A_0^h)^{-1}R^hD^h||_{1,0},
\end{aligned}$$

where D^h stands for αA_1^h, αA_2^h, and βB^h. Thus we obtain

$$I^h(M^h)^{-1}R^hD^h \overset{c.c}{\to} M^{-1}D.$$

On the basis of collectively compact theory [1], we can find an infinite subsequence in $\{I^h(M^h)^{-1}R^h\hat{A}_1^hz_h\}$ which converges as $h \to 0$. Without loss of generality, we will use the same notation for this convergent subsequence that we used for the original sequence. Similarly, the second and third terms in (2.28) have the same result. Hence, there exists an infinite subsequence $H_1 \subset H$ such that (2.28) converges. Correspondingly, there exists an infinite subsequence $H_2 \subset H_1 \subset H$ such that the second component of $I^h(M^h)^{-1}R^h(\alpha A_1^h + \alpha A_2^h + \beta B^h)R^hz_h$ converges. As above, there exists an infinite subsequence $H_3 \subset H_2 \subset H_1 \subset H$ such that $\{I^h(M^h)^{-1}R^h(\beta B^h)R^hz_h, h \in H_3\}$ converges. This shows that $\{\hat{L}^h\}$ is a collectively compact sequence, and $\hat{L}^h$ is pointwisely convergent to L. ∎

COROLLARY 2.7. *Assuming that* (1.2) *has a unique solution and h is sufficiently small, then there exists a unique solution $\hat{v}^h$ in* (2.26)*, and $\hat{v}^h$ has the following error bound under the norm of $C[0, 2\pi)$:*

$$||\hat{v}^h - v|| \le ||(I + L)^{-1}||\frac{||(\hat{L}^h - L)\hat{F}|| + ||(\hat{L}^h - L)\hat{L}^hv||}{1 - ||(I + \hat{L}^h)^{-1}(\hat{L}^h - L)\hat{L}^h||}. \tag{2.29}$$

3. Asymptotic expansions of errors and EAs. In this section, we derive the asymptotic expansion of the solution errors and describe the EA.

THEOREM 3.1. *Assume that there exists a unique solution in* (1.2a)*, F, F^h are computed by* (2.3) *and* (2.7) *respectively, and $x_i(t)$, $g(t) \in C^6[0, 2\pi)$(i = 1, 2). Then there exists a function $\varpi \in C^6[0, 2\pi)$ independent of h such that*

$$(v - \hat{v}^h)|_{t=t_j} = h^3\varpi|_{t=t_j} + O(h^5). \tag{3.1}$$

Proof. By the quadrature rule (2.5) and (2.6), there exists the asymptotic expansion [10, 11]

$$(F - F^h)|_{t=t_i} = h^3 I^h R^h \psi|_{t=t_i} + O(h^5), \tag{3.2}$$

with $\psi = -\eta\xi'(-2)g''(t)/\pi$ and

$$\eta = \begin{cases} 0 \text{ as } \beta = 0, \\ 1 \text{ as } \beta \neq 0. \end{cases}$$

Using (2.3), (2.5), and (2.6), we can easily obtain

$$\begin{aligned}
&(\beta\theta(y)E^h + \alpha A_0^h + \alpha A_1^h + \alpha A_2^h + \beta B^h)R^h(\hat{v}^h - v)|_{t=t_i} \\
&= F^h - I^h(\beta\theta(y)E^h + \alpha A_0^h + \alpha A_1^h + \alpha A_2^h + \beta B^h)R^h v|_{t=t_i} \\
&= F^h - [((\beta\theta(y)E + \alpha A_0 + \alpha A_1 + \alpha A_2 + \beta B)v - h^3 I^h R^h \gamma)|_{t=t_i} + O(h^5)] \\
&= (\hat{F}^h - F)|_{t=t_i} + h^3 I^h R^h \gamma|_{t=t_i} + O(h^5) \\
&= h^3 I^h R^h \varphi|_{t=t_i} + O(h^5),
\end{aligned}$$

where $\gamma = \alpha\xi'(-2)v''(t)/\pi$, and $\varphi = \psi + \gamma$. From Theorem 2.6, we get

$$(E^h + \hat{L}^h)(v - \hat{v}^h)|_{t=t_i} = h^3(\hat{M}^h)^{-1} I^h R^h \varphi|_{t=t_i} + O(h^5). \tag{3.3}$$

Define the auxiliary equation

$$(E + L)\varpi = M^{-1}\varphi \tag{3.4a}$$

and its approximate equation

$$(E^h + \hat{L}^h)\varpi^h = (\hat{M}^h)^{-1} I^h R^h \varphi. \tag{3.4b}$$

Substituting (3.4) into (3.3) yields

$$(E^h + \hat{L}^h)(v - \hat{v}^h - h^3\varpi^h)|_{t=t_i} = O(h^5). \tag{3.5}$$

Since $(E^h + \hat{L}^h)^{-1}$ is uniformly bounded by Theorem 2.6, we get

$$(v - \hat{v}^h - h^3\varpi^h)|_{t=t_i} = O(h^5). \tag{3.6}$$

Replacing ϖ^h in (3.6) with ϖ and applying Theorem 2.6, we complete the proof. ∎

The asymptotic expansions (3.1) imply that the EA can be applied to solve (1.2). Moreover, the high order $O(h^5)$ of accuracy can be obtained on coarse grids of $\partial\Omega$ in parallel. The EA can be described as follows [10, 11, 15, 23]:

Step 1. Take h and $h/2$, and solve (2.7a) in parallel, where $v^h(t_i)$ are their solutions.

Step 2. Compute the solutions at the coarse grid points

$$v^*(t_i) = \frac{1}{7}[8v^{h/2}(t_i) - v^h(t_i)], \tag{3.7}$$

and $u(y)$ $(y \in \Omega')$ can then be obtained by (2.8).

Moreover, using $|v^*(t_i) - v(t_i)| = O(h^5)$, we obtain the a posteriori estimate

$$\begin{aligned}
&|v(t_i) - v^{h/2}(t_i)| \\
&\leq \left|v(t_i) - \frac{1}{7}[8v^{h/2}(t_i) - v^h(t_i)]\right| + |v^{h/2}(t_i) - v^*(t_i)| \\
&\leq \frac{8}{7}|v^{h/2}(t_i) - v^h(t_i)| + O(h^5).
\end{aligned} \tag{3.8}$$

4. Numerical examples.

Example 4.1 (see [14]). We consider the scattering of a plane wave u^i by a sound-soft cylinder with a nonconvex boomerang-shaped cross section with boundary $\partial\Omega$ and described by the parametric representation

$$x(t) = (\cos t + 0.65\cos 2t - 0.65, 1.5\sin t), \quad 0 \le t \le 2\pi.$$

The incident wave is given by $u^i(x) = e^{ikd\cdot x}$, where d denotes a unit vector giving the direction of propagation. For the scattered wave u we have to solve an exterior Dirichlet problem with boundary values $g = -u^i$ on $\partial\Omega$. The far-field pattern u_∞ is defined by the asymptotic behavior of the scattered wave

$$u(x) = \frac{e^{ik|x|}}{\sqrt{|x|}}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty,$$

uniformly in all directions $\hat{x} = x/|x|$. From the asymptotic formula

$$H_0^{(1)}(z) = \sqrt{\frac{2}{\pi z}}e^{i(z-\pi/4)}\left\{1 + O\left(\frac{1}{z}\right)\right\}, \quad z \to \infty,$$

for the Hankel function, we see that the far-field pattern of the single-layer potential (1.4) is given by $u_\infty(\hat{x}) = c\int_{\partial\Omega} e^{-ikx\cdot y}v(y)ds(y)$, where $c = -e^{i\pi/4}/\sqrt{2\pi k}$.

When $d = (1, 0)$, Tables 1 and 2 list some approximate values for the far-field patterns $u_\infty(d)$ and $u_\infty(-d)$ in the forward direction d and backward direction $-d$ as the wave number $k = 1$, respectively. Let the number of nodes be $n = 2\pi/h$, the real part of the error be $\mathrm{Re}(e_\infty(d))$, and the imaginary part be $\mathrm{Im}(e_\infty(d))$. $\mathrm{Re}(e_\infty^E(d))$ and $\mathrm{Im}(e_\infty^E(d))$ show the error after using the EA once. Note that the fast convergence is clearly obtained.

Table 1
Errors for $u_\infty(d)$ in the forward direction d when $k = 1$.

n	$\mathrm{Re}(e_\infty(d))$	$\mathrm{Im}(e_\infty(d))$	$\mathrm{Re}(e_\infty^E(d))$	$\mathrm{Im}(e_\infty^E(d))$
2^3	3.19e-2	3.63e-3	4.87e-4	1.47e-4
2^4	3.57e-3	3.26e-4	1.20e-4	1.59e-5
2^5	4.35e-4	2.74e-5	5.64e-7	1.09e-7
2^6	5.39e-5	3.32e-6	1.81e-8	4.28e-9
2^7	6.73e-6	4.12e-7	5.33e-10	1.96e-10
2^8	8.41e-7	5.13e-8		

Table 2
Errors for $u_\infty(-d)$ in the backward direction d when $k = 1$.

n	$\mathrm{Re}(e_\infty(-d))$	$\mathrm{Im}(e_\infty(-d))$	$\mathrm{Re}(e_\infty^E(-d))$	$\mathrm{Im}(e_\infty^E(-d))$
2^3	1.99e-2	9.28e-3	2.05e-4	1.60e-4
2^4	7.33e-4	5.05e-4	9.71e-7	1.71e-6
2^5	5.54e-5	1.26e-4	1.71e-8	1.71e-7
2^6	6.91e-6	1.56e-5	6.57e-9	1.14e-8
2^7	8.64e-7	1.95e-6	7.61e-11	1.08e-10
2^8	1.08e-7	2.44e-7		

Table 3 lists the maximum and minimum values of the eigenvalue moduli $|\lambda^h|$ of the discretization matrix, which gives

$$\frac{\mathrm{Cond.}|_{n=2^{m+1}}}{\mathrm{Cond.}|_{n=2^m}} \approx 2 \quad (m = 3, \ldots, 7).$$

TABLE 3
Maximum and minimum eigenvalue moduli.

n	2^3	2^4	2^5	2^6	2^7	2^8
λ_{max}	1.436	1.439	1.438	1.438	1.438	1.438
λ_{min}	0.296	0.176	8.69e-2	4.34e-2	2.17e-2	1.08e-2

TABLE 4
Error e^h, error ratio r^h, and error e_E^h for $k = 1$, $q = 2$.

t	$t = 0$			$t = \pi/2$			$t = \pi$		
n	e^h	r^h	e_E^h	e^h	r^h	e_E^h	e^h	r^h	e_E^h
2^3	3.05e-2	9.56	7.01e-4	9.56e-3	37.0	1.07e-3	9.59e-3	62.2	1.54e-3
2^4	3.19e-3	8.71	3.85e-5	2.58e-4	8.92	3.71e-6	1.54e-4	6.38	5.78e-6
2^5	3.66e-4	8.15	8.80e-7	2.89e-5	7.96	1.08e-8	2.44e-5	8.00	1.45e-9
2^6	4.49e-5	8.01	2.58e-8	3.63e-6	8.00	3.76e-10	3.05e-6	8.00	2.21e-11
2^7	5.60e-6	8.01	7.26e-10	4.54e-7	7.99	3.29e-11	3.81e-7	7.99	1.28e-11
2^8	6.99e-7	8.00	1.19e-11	5.68e-8	8.00	1.16e-11	4.77e-8	8.00	7.19e-12
2^9	8.74e-8			7.11e-9			5.96e-9		

TABLE 5
Error e^h, error ratio r^h, and error e_E^h for $k = 5$, $q = 2$.

t	$t = 0$			$t = \pi/2$			$t = \pi$		
n	e^h	r^h	e_E^h	e^h	r^h	e_E^h	e^h	r^h	e_E^h
2^3	2.99e-1	21.51	5.87e-2	1.37e-1	15.30	2.99e-2	3.54e-1	136.15	5.35e-2
2^4	1.39e-2	8.52	1.16e-4	8.95e-3	10.99	3.48e-4	2.60e-3	18.57	2.11e-4
2^5	1.63e-3	8.35	1.01e-5	8.14e-4	8.30	4.36e-6	1.40e-4	8.00	1.15e-7
2^6	1.95e-4	8.09	3.04e-7	9.80e-5	8.30	1.30e-7	1.74e-5	8.01	4.04e-9
2^7	2.41e-5	8.00	9.38e-9	1.21e-5	8.10	3.99e-9	2.17e-6	7.98	1.18e-10
2^8	3.01e-6	8.00	2.68e-10	1.51e-6	7.99	1.06e-10	2.72e-7	8.00	3.33e-12
2^9	3.76e-7			1.89e-7			3.40e-8		

This coincides with Lemma 2.1 perfectly, where Cond. is the condition number.

Example 4.2 (see [13]). We consider the case in which the exact solution to the Neumann problem is given by $u(x) = N_0(k|x - x_0|)$, where $x_0 = (q, 0)$ with $q > 1$. Then the right-hand side is given by $\tilde{g}(t) = -k\frac{1-q\cos t}{c(t)}N_1(kc(t))$ and the exact solution of the integral equation by $v(t) = N_0(kc(t))$, where $c(t) = 1 + q^2 - 2q\cos t$.

Tables 4 and 5 accord with what we expect in our general error analysis, where the number of nodes n is $2\pi/h$; $e^h(t) = |u(t) - u^h(t)|$ gives the absolute error at a given point t; $r^h(t) = e^h(t)/e^{h/2}(t)$ shows the convergence ratio; and $e_E^h(t)$ shows the error at the given point after using the EA once.

Let $\|e^h\|_\infty = \|u-u^h\|_\infty$ and $\|e_E^h\|_\infty = \|u-u_E^h\|_\infty$ denote the maximum error over all nodes. By using linear regression, we get $\|e^h\|_\infty = 0.057h^{3.06}$, $\|e_E^h\|_\infty = 0.0045h^{5.1}$ in the case of $k = 1$, and $\|e^h\|_\infty = 0.58h^{3.29}$, $\|e_E^h\|_\infty = 0.13h^{5.43}$ when $k = 5$, which verifies Theorem 3.1 numerically. Values are shown in Figure 1.

5. Conclusions. The numerical results in the previous section accord with our theoretical analysis and support the claim of high performance of the MQM and the effectiveness of the EA for higher accuracy. Since the discretization matrix of the BIE is full, the larger the scale of the problem, the higher the precision that the EA attains.

Generally, there are two main advantages of the MQM: (1) evaluating each element of discretization matrices is very simple and straightforward, which does not

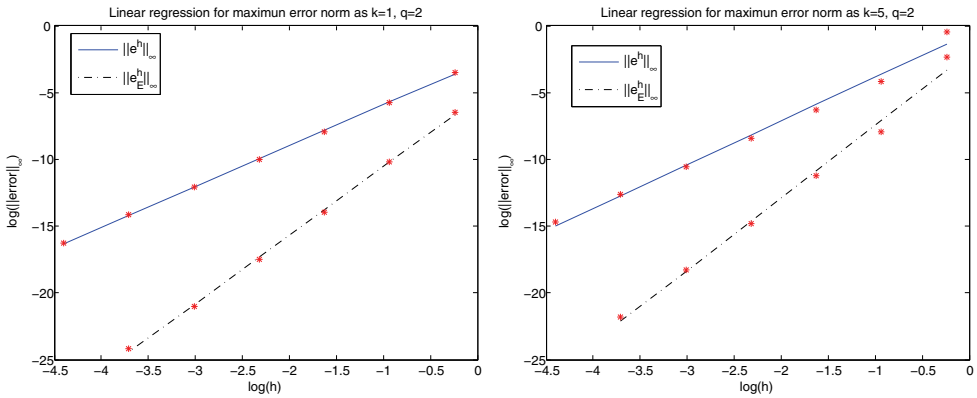

FIG. 1. *Linear regression analysis.*

require any singular integrals; (2) the algorithm has a high accuracy of $O(h^3)$. However, the analysis of the MQM is more challenging than those of the Galerkin and the collocation methods because its theory is no longer within the framework of the projection theory.

In this paper we discussed the MQM and EA only for problems with a smooth boundary $\partial\Omega$. The discussion of problems with a nonsmooth boundary will be presented in a separate paper.

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REFERENCES

[1] P. M. Anselone, *Collectively Compact Operator Approximation Theory*, Prentice–Hall, Englewood Cliffs, NJ, 1971.
[2] K. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, Cambridge, UK, 1997.
[3] K. Atkinson, *The numerical solution of Laplace's equation on a wedge*, IMA J Numer. Anal., 4 (1984), pp. 19–41.
[4] G. A. Chandler, *Galerkin's method for boundary integral equations on polygonal domains*, J. Austral. Math Soc. Ser. B, 26 (1984), pp. 1–13.
[5] F. Chatelin, *Spectral Approximation of Linear Operator*, Academic Press, New York, 1983.
[6] D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory*, John Wiley, New York, 1983.
[7] Z. Daniel, *Standard Mathematical Tables and Formulae*, CRC Press, New York, 1996.
[8] P. J. Davis, *Methods of Numerical Integration*, Academic Press, Orlando, FL, 1984.
[9] A. S. Householder, *The Theory of Matrices in Numerical Analysis*, Blaisdell, New York, 1964.
[10] J. Huang and T. Lu, *The mechanical quadrature methods and their extrapolation for solving BIE of Steklov eigenvalue problems*, J. Comput. Math., 22 (2004), pp. 719–726.
[11] J. Huang and T. Lu, *Splitting extrapolations for solving boundary integral equations of linear elasticity Dirichlet problems on polygons by mechanical quadrature methods*, J. Comput. Math., 24 (2006), pp. 9–18.
[12] J. Huang and T. Lu, *The mechanical quadrature methods and their splitting extrapolations for solving first-kind boundary integral equations on polygonal regions*, Math. Numer. Sin., 26 (2004), pp. 51–60 (in Chinese).
[13] R. Kress, *Linear Integral Equations*, Springer-Verlag, Berlin, 1989.
[14] R. Kress and I. H. Sloan, *On the numerical solution of a logarithmic integral equation of the first kind for the Helmholtz equation*, Numer. Math., 66 (1993), pp. 199–214.

[15] C. B. Liem, T. Lü, and T. M. Shih, *The Splitting Extrapolation Method*, World Scientific, Singapore, 1995.

[16] F. Paris and J. Cans, *Boundary Element Method*, Oxford University Press, New York, 1997.

[17] U. Rüde and A. Zhou, *Multi-parameter extrapolation methods for boundary integral equations*, Adv. Comput. Math., 9 (1998), pp. 173–190.

[18] J Shen and L.-L. Wang, *Spectral approximation of the Helmholtz equation with high wave numbers*, SIAM J. Numer. Anal., 43 (2005), pp. 623–644.

[19] A. Sidi and M. Israeli, *Quadrature methods for periodic singular Fredholm integral equation*, J. Sci. Comput., 3 (1988), pp. 201–231.

[20] I. H. Sloan, *A quadrature-based approach to improving the collocation method*, Numer. Math., 54 (1988), pp. 41–56.

[21] Y. S. Xu and Y. H. Zhao, *An extrapolation method for a class of boundary integral equations*, Math. Comp., 65 (1996), pp. 587–610.

[22] Y. Yan, *A fast boundary element method for the two-dimensional Helmholtz equation*, Comput. Methods Appl. Mech. Engrg., 110 (1993), pp. 285–299.

[23] R. Zhu, J. Huang, and T. Lu, *Mechanical quadrature methods and their splitting extrapolations for solving boundary integral equations of axisymmetric Laplace mixed boundary value problems*, Eng. Anal. Boundary Elem., 30 (2006), pp. 391–398.