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Non-iterative domain decomposition methods for a non-stationary Stokes–Darcy model with Beavers–Joseph interface condition ☆

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ABSTRACT

In order to solve a non-stationary Stokes–Darcy model with Beavers–Joseph interface condition, two non-iterative domain decomposition methods are proposed. At each time step, results from previous time steps are utilized to approximate the information on the interface and decouple the two physics. Both of the two methods are parallel. Numerical results suggest that the first method has accuracy order $O(h^3 + \Delta t)$. In order to improve the accuracy and efficiency, a three-step backward differentiation is used in the second method to achieve an accuracy order $O(h^3 + \Delta t^3)$, which is illustrated by a numerical example.

1. Introduction

Domain decomposition is a natural and efficient way to solve a partial differential equation in parallel. Its major difficulty is to define the values on the interface between subdomains. For elliptic equations, convergent iterations are used to predict the values we need on the interfaces and approximate the solutions based on an initial guess [12,50,72,83,92]. For time-dependent problems, there are two popular ways for domain decomposition. The first one is traditional, which is to apply the iterative domain decomposition method for elliptic equations at each time step [16,28,30,41,67,71,82]. The second one is to take advantage of information gained in the previous time steps to construct a non-iterative domain decomposition method, such as the explicit/implicit domain decomposition (EIDD) method [32–34,43,96,97], stabilized EIDD method [98,99], IPIC method [62], ADI method [61], and others [42,95]. Based on an implicit discretization in time, the second framework makes use of the results of the previous time steps to predict the values on the interface at the current time step. Obviously the second way saves on both computation and communication costs because it is non-iterative. The key issue about non-iterative domain decomposition is how to obtain optimal accuracy and better stability because it uses lagged results from the previous time steps, i.e., and explicit treatment, instead of iterations to predict the interface values.

The Stokes–Darcy model has been studied for many interesting problems, such as surface water and groundwater flows, oil flows in a vuggy porous medium, groundwater system in karst aquifers, and industrial filtrations, etc.; see [2,18,19,22,29,35,44,53,57,69,75,78,90] and references cited therein. This model uses two different systems of partial differential equations to govern the free flow and the porous media flow separately and then couples them together through some interface conditions, enabling a better description of the physics compared to that possible with a single-system model. Therefore, different methods have been developed for solving the Stokes–Darcy model, such as coupled finite element methods [2,4,18–20,23,44,46,64,86,90], domain decomposition methods (DDM) [17,24,35–40,56,59], Lagrange multiplier methods [3,51,52,69], two grid method [15,75], discontinuous Galerkin methods [21,26,53,63,84,85], and many others

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[10,11,27,45,47–49,65,68,70,76,77,80,89]. Meanwhile Stokes-Brinkman and other similar models are also studied and compared, see [1,5,8–10,13,14,25,31,54,73,74,79,81,88,91,93,94] and references cited therein.

Among these methods, physics-based domain decomposition is more natural than others since the problem domain naturally consists of two different sub-domains. The possibility of parallel computation and the nature preconditioning of domain decomposition methods have motivated the development of some efficient methods for solving the system of discrete equations; see [17,24,38–40,59]. On the other hand, most of previous works on the Stokes–Darcy system use the Beavers–Joseph–Saffman–Jones (BJSJ) [58,60,87] interface conditions or even further simplification because well-posedness can be demonstrated in a fairly straightforward manner. However, the BJSJ condition ignores certain contributions made by the flow in the porous media flow to the coupling of the two models; the ignored contributions may be important [22] in some applications such as karst aquifers. The more physically faithful Beavers–Joseph (BJ) [7] interface condition is more accurate because it fully accounts for the contributions of the two flows in the coupling of the two models. Additionally, most of the existing work is for the steady state Stokes–Darcy model instead of the time-dependent one, which is more interesting. In this article, we develop two parallel non-iterative domain decomposition methods for the non-stationary Stokes–Darcy model with Beavers–Joseph interface condition.

The rest of paper is organized as follows. In Section 2, we introduce the Stokes–Darcy system with Beavers–Joseph interface condition. In Section 3, the system is decoupled by using some Robin boundary conditions. In Section 4, the semi-discretization of the decoupled system is presented. Then two parallel non-iterative domain decomposition methods are proposed with a numerical example in Sections 5 and 6 separately.

2. Stokes–Darcy model

We consider a coupled Stokes–Darcy system on a bounded domain $\Omega = \Omega_m \bigcup \Omega_c \subset \mathbb{R}^{\mathbf{d}}$, $(\mathbf{d} = 2, 3)$, see Fig. 1. In the porous media region Ω_m, let $\vec{u}_m$ denote the fluid discharge rate in the porous media, $\mathbb{K}$ denote the hydraulic conductivity tensor, f_m denote the sink/source term and ϕ_m denote the hydraulic head. Specifically, $\phi_m = z + \frac{p_m}{\rho g}$ where p_m is the dynamic pressure, z is the height, ρ is the density and g is the gravity constant. Then the porous media flow is assumed to satisfy the following Darcy equations.

$$\vec{u}_m = -\mathbb{K}\nabla\phi_m, \tag{2.1}$$

$$\frac{\partial\phi_m}{\partial t} + \nabla\cdot\vec{u}_m = f_m, \quad t \in [t_0, T]. \tag{2.2}$$

Eliminating $\vec{u}_c$, we obtain the second-order form of the Darcy system

$$\frac{\partial\phi_D}{\partial t} - \nabla\cdot(\mathbb{K}\nabla\phi_D) = f_D. \tag{2.3}$$

In the fluid region Ω_c, let $\vec{u}_c$ denote the fluid velocity, p_c denote the kinematic pressure, $\vec{f}_c$ denote the external body force, and v denote the kinematic viscosity of the fluid. Additionally, $\mathbb{T}(\vec{u}_c, p_c) = 2v\mathbb{D}(\vec{u}_c) - p_c\mathbb{I}$ is the stress tensor and $\mathbb{D}(\vec{u}_c) = 1/2(\nabla\vec{u}_c + \nabla^T\vec{u}_c)$ is the deformation tensor. Then the fluid flow is assumed to satisfy the Stokes equations

$$\frac{\partial\vec{u}_c}{\partial t} - \nabla\cdot\mathbb{T}(\vec{u}_c, p_c) = \vec{f}_c, \quad t \in [t_0, T], \tag{2.4}$$

$$\nabla\cdot\vec{u}_c = 0. \tag{2.5}$$

Let $\Gamma = \overline{\Omega}_m \cap \overline{\Omega}_c$ denote the interface between the fluid and porous media regions. Along the interface Γ, we first impose the following two well-accepted interface conditions:

$$\vec{u}_c\cdot\vec{n}_c = -\vec{u}_m\cdot\vec{n}_m, \tag{2.6}$$

$$-\vec{n}_c\cdot(\mathbb{T}(\vec{u}_c, p_c)\cdot\vec{n}_c) = g(\phi_m - z), \tag{2.7}$$

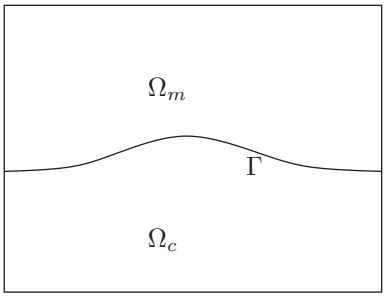

Fig. 1. A sketch of the porous median domain Ω_m, fluid domain Ω_c, and the interface Γ.

where $\vec{n}_c$ and $\vec{n}_m$ denote the unit outer normal to the fluid and the porous media regions at the interface Γ, respectively. These two interface conditions are for the continuity of normal velocity and the balance of force normal to the interface. Since both of them are in normal directions, the following Beavers–Joseph (BJ) interface condition [7] is imposed in the tangential direction on the interface.

$$-\boldsymbol{\tau}_j \cdot (\mathbb{T}(\vec{u}_c, p_c) \cdot \vec{n}_c) = \frac{\alpha \nu \sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}} \boldsymbol{\tau}_j \cdot (\vec{u}_c - \vec{u}_m), \tag{2.8}$$

where $\boldsymbol{\tau}_j$ $(j = 1, \ldots, d-1)$ denote mutually orthogonal unit tangential vectors to the interface Γ, and $\prod = \frac{\mathbb{K}\nu}{g}$.

Assume that the hydraulic head ϕ_m and the fluid velocity $\vec{u}_c$ satisfies homogeneous Dirichlet boundary condition except on Γ, i.e., $\phi_m = 0$ on the boundary $\partial\Omega_m \setminus \Gamma$ and $\vec{u}_c = 0$ on the boundary $\partial\Omega_c \setminus \Gamma$. Assume that the hydraulic head ϕ_m and the fluid velocity $\vec{u}_c$ satisfies the following initial conditions.

$$\phi_m(0, x, y) = \phi_0(x, y), \tag{2.9}$$
$$\vec{u}_c(0, x, y) = \vec{u}_0(x, y). \tag{2.10}$$

Then the spaces that we utilize are

$$\begin{aligned}
&X_c = \{\vec{v} \in [H^1(\Omega_c)]^d | \vec{v} = 0 \quad \text{on } \partial\Omega_c \setminus \Gamma\},\\
&Q_c = L^2(\Omega_c),\\
&X_m = \{\psi \in H^1(\Omega_m) | \psi = 0 \quad \text{on } \partial\Omega_m \setminus \Gamma\},\\
&L^2(t_0, T; Q_c) = \{\phi : \phi(t, \cdot) \in Q_c, \quad \forall t \in [t_0, T]\},\\
&H^1(t_0, T; X_m, X_m') = \{\phi : \phi(t, \cdot) \in X_m \quad \text{and} \quad \frac{\partial \phi}{\partial t}(t, \cdot) \in X_m', \quad \forall t \in [t_0, T]\},\\
&H^1(t_0, T; X_c, X_c') = \{\phi : \phi(t, \cdot) \in X_c \quad \text{and} \quad \frac{\partial \phi}{\partial t}(t, \cdot) \in X_c', \quad \forall t \in [t_0, T]\}.
\end{aligned}$$

where X_m' and X_c' are the dual spaces of X_m and X_c. For the domain D ($D = \Omega_c$ or Ω_m), $(\cdot, \cdot)_D$ denotes the L^2 inner product on the domain D, and $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product on the interface Γ or the duality pairing between $(H_{00}^{1/2}(\Gamma))'$ and $H_{00}^{1/2}(\Gamma)$. P_τ denotes the projection onto the tangent space on Γ, i.e.

$$P_\tau \vec{u} = \sum_{j=1}^{d-1} (\vec{u} \cdot \boldsymbol{\tau}_j) \boldsymbol{\tau}_j.$$

We also define the following bilinear forms

$$\begin{aligned}
&a_m(\phi_m, \psi) = (\mathbb{K}\nabla\phi_m, \nabla\psi)_{\Omega_m},\\
&a_c(\vec{u}_c, \vec{v}) = 2\nu(\mathbb{D}(\vec{u}_c), \mathbb{D}(\vec{v}))_{\Omega_c},\\
&b_c(\vec{v}, q) = -(\nabla \cdot \vec{v}, q)_{\Omega_c}.
\end{aligned}$$

With these notations, the weak formulation of the coupled Stokes–Darcy problem is given as follows [18,19]: find $(\vec{u}_c, p_c) \in H^1(t_0, T; X_c, X_c') \times L^2(t_0, T; Q_c)$ and $\phi_m \in H^1(t_0, T; X_m, X_m')$ such that

$$\begin{aligned}
&\left(\frac{\partial \vec{u}_c}{\partial t}, \vec{v}\right)_{\Omega_c} + \eta\left(\frac{\partial \phi_m}{\partial t}, \psi\right)_{\Omega_m} + a_c(\vec{u}_c, \vec{v}) + b_c(\vec{v}, p_c) + \eta a_m(\phi_m, \psi) + \langle g\phi_m, \vec{v} \cdot \vec{n}_c \rangle - \eta\langle \vec{u}_c \cdot \vec{n}_c, \psi \rangle\\
&\quad + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}} \langle P_\tau(\vec{u}_c + \mathbb{K}\nabla\phi_m), P_\tau \vec{v} \rangle\\
&\quad = \eta(f_m, \psi)_{\Omega_m} + (\vec{f}_c, \vec{v})_{\Omega_c} + \langle gz, \vec{v} \cdot \vec{n}_c \rangle, \quad \forall \vec{v} \in X_c, \ \psi \in X_m,
\end{aligned} \tag{2.11}$$

$$b_c(\vec{u}_c, q) = 0, \quad \forall q \in Q_c, \tag{2.12}$$

It is easy to see that the system of (2.11) and (2.12) is well-posed for $\vec{f}_c \in [L^2(\Omega_c)]^d$ and $f_m \in L^2(\Omega_m)$ for large enough scaling constant η [19].

3. Robin boundary conditions and the decoupled system

In order to solve the coupled Stokes–Darcy problem utilizing the domain decomposition idea, we naturally consider Robin boundary conditions for the Darcy and the Stokes equations by following the idea in [17,24].

First let us propose the following Robin condition for the Darcy system

$$\mathbb{K}\nabla\hat{\phi}_m \cdot \vec{n}_m + g\hat{\phi}_m = \xi_m \quad \text{on } \Gamma \tag{3.13}$$

for a given function ξ_m defined on Γ. Hence, the corresponding weak formulation for the Darcy system is given by: for $\xi_m \in L^2(t_0,T;L^2(\Gamma))$, find $\hat{\phi}_m \in H^1(t_0,T;X_m,X_m')$ such that

$$\left(\frac{\partial\hat{\phi}_m}{\partial t},\psi\right)_{\Omega_m} + a_m(\hat{\phi}_m,\psi) + \left\langle g\hat{\phi}_m,\psi\right\rangle = (f_m,\psi)_{\Omega_m} + \langle\xi_m,\psi\rangle, \quad \forall\psi\in X_m, \tag{3.14}$$

$$\hat{\phi}_m(t_0) = \phi_0. \tag{3.15}$$

On the other hand, we consider the following two Robin type conditions for the Stokes equations

$$\vec{n}_c\cdot(\mathbb{T}(\hat{\vec{u}}_c,\hat{p}_c)\cdot\vec{n}_c) + \hat{\vec{u}}_c\cdot\vec{n}_c = \xi_c \quad \text{on } \Gamma, \tag{3.16}$$

$$-P_\tau(\mathbb{T}(\hat{\vec{u}}_c,p_c)\cdot\vec{n}_c) - \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}P_\tau\hat{\vec{u}}_c = \vec{\xi}_{c\tau} \quad \text{on } \Gamma \tag{3.17}$$

for a given function ξ_c defined on Γ. Then, the corresponding weak formulation for the Stokes system is given by: for $\xi_c \in L^2(t_0,T;L^2(\Gamma))$, find $\hat{\vec{u}}_c \in H^1(t_0,T;X_c,X_c')$ and $\hat{p}_c \in L^2(t_0,T;Q_c)$ such that

$$\begin{aligned}&\left(\frac{\partial\hat{\vec{u}}_c}{\partial t},\vec{v}\right)_{\Omega_c} + a_c(\hat{\vec{u}}_c,\vec{v}) + b_c(\vec{v},\hat{p}_c) + \left\langle\hat{\vec{u}}_c\cdot\vec{n}_c,\vec{v}\cdot\vec{n}_c\right\rangle + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\left\langle P_\tau\hat{\vec{u}}_c,P_\tau\vec{v}\right\rangle\\ &\quad = (\vec{f}_c,\vec{v})_{\Omega_c} + \langle\xi_c,\vec{v}\cdot\vec{n}_c\rangle - \left\langle\vec{\xi}_{c\tau},P_\tau\vec{v}\right\rangle, \quad \forall\vec{v}\in X_c,\end{aligned} \tag{3.18}$$

$$b_c(\hat{\vec{u}}_c,q) = 0, \quad \forall q\in Q_c, \tag{3.19}$$

$$\hat{\vec{u}}_c(t_0) = \vec{u}_0. \tag{3.20}$$

The Stokes and Darcy systems with Robin boundary conditions can be combined into one system. Indeed, it is easy to see that if ξ_m and ξ_c are given, then, there exists a unique solution $(\hat{\phi}_m,\hat{\vec{u}}_c,\hat{p}_c) \in H^1(t_0,T;X_m,X_m')\times H^1(t_0,T;X_c,X_c')\times L^2(t_0,T;Q_c)$ such that

$$\begin{aligned}&\left(\frac{\partial\hat{\vec{u}}_c}{\partial t},\vec{v}\right)_{\Omega_c} + \eta\left(\frac{\partial\hat{\phi}_m}{\partial t},\psi\right)_{\Omega_m} + a_c(\hat{\vec{u}}_c,\vec{v}) + b_c(\vec{v},\hat{p}_c) + \eta a_m(\hat{\phi}_m,\psi) + \left\langle\hat{\vec{u}}_c\cdot\vec{n}_c,\vec{v}\cdot\vec{n}_c\right\rangle + \eta\left\langle g\hat{\phi}_m,\psi\right\rangle\\ &\quad + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\left\langle P_\tau\hat{\vec{u}}_c,P_\tau\vec{v}\right\rangle = \eta(f_m,\psi)_{\Omega_m} + (\vec{f}_c,\vec{v})_{\Omega_c} + \langle\xi_c,\vec{v}\cdot\vec{n}_c\rangle + \eta\langle\xi_m,\psi\rangle - \left\langle\vec{\xi}_{c\tau},P_\tau\vec{v}\right\rangle, \quad \forall\psi\in Q_c,\ \ \vec{v}\in X_c,\end{aligned} \tag{3.21}$$

$$b_c(\hat{\vec{u}}_c,q) = 0, \quad \forall q\in Q_c, \tag{3.22}$$

$$\hat{\phi}_m(t_0) = \phi_0, \hat{\vec{u}}_c(t_0) = \vec{u}_0. \tag{3.23}$$

Then we obtain the following compatibility lemma whose proof is similar to that of Lemma 2.2 in [24].

Lemma 3.1. *Let $(\phi_m,\vec{u}_c,p_c)$ be the solution of the coupled Stokes–Darcy system (2.11) and (2.12) and let $(\hat{\phi}_m,\hat{\vec{u}}_c,\hat{p}_c)$ be the solution of the decoupled Stokes and Darcy system with Robin boundary conditions at the interface (3.21) and (3.22). Then, $(\hat{\phi}_m,\hat{\vec{u}}_c,\hat{p}_c) = (\phi_m,\vec{u}_c,p_c)$ if and only if $\xi_c,\vec{\xi}_{c\tau}$, and ξ_m satisfy the following compatibility conditions:*

$$\xi_m = \hat{\vec{u}}_c\cdot\vec{n}_c + g\hat{\phi}_m, \tag{3.24}$$

$$\xi_c = \hat{\vec{u}}_c\cdot\vec{n}_c - g\hat{\phi}_m + gz, \tag{3.25}$$

$$\vec{\xi}_{c\tau} = \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{trace(\prod)}}P_\tau(\mathbb{K}\nabla\hat{\phi}_m). \tag{3.26}$$

4. Semi-discretization of the decoupled system

Now we present the semi-discretization for (3.14), (3.18) and (3.19). Suppose we have finite element spaces $X_{Dh}\subset X_m$, $X_{Sh}\subset X_c$ and $Q_{Sh}\subset Q_c$. Here we assume that $X_{Sh}\subset X_c$ and $Q_{Sh}\subset Q_c$ satisfy the following inf–sup condition: there exists a constant $\gamma>0$

$$\inf_{0\neq q\in Q_{sh}}\sup_{0\neq\vec{v}\in X_{Sh}}\frac{b_c(\vec{v},q)}{\|\vec{v}\|_1\|q\|_0} > \gamma. \tag{4.27}$$

Define $P_h : X_m \to X_{Dh}$ and $\mathbb{P}_h : X_c \to X_{Sh}$ to be the regular orthogonal projections. With finite element approximation in space, we can approximate (3.14), (3.18) and (3.19) as follows: Find $\hat{\phi}_h \in H^1(t_0, T; X_{Dh})$, $\hat{\vec{u}}_h \in H^1(t_0, T; X_{Sh})$ and $\hat{p}_h \in L^2(t_0, T; Q_{Sh})$ such that

$$\left(\frac{\partial \hat{\phi}_h}{\partial t}, \psi_h\right)_{\Omega_m} + a_m(\hat{\phi}_h, \psi_h) + \left\langle g\hat{\phi}_h, \psi_h \right\rangle = (f_m, \psi_h)_{\Omega_m} + \left\langle \hat{\vec{u}}_h \cdot \vec{n}_c + g\hat{\phi}_h, \psi_h \right\rangle, \quad \forall \psi_h \in X_{Dh}, \tag{4.28}$$

$$\begin{aligned}&\left(\frac{\partial \hat{\vec{u}}_h}{\partial t}, \vec{v}_h\right)_{\Omega_c} + a_c(\hat{\vec{u}}_h, \vec{v}_h) + b_c(\vec{v}_h, \hat{p}_h) + \left\langle \hat{\vec{u}}_h \cdot \vec{n}_c, \vec{v}_h \cdot \vec{n}_c \right\rangle + \frac{\alpha v\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} \left\langle P_\tau \hat{\vec{u}}_h, P_\tau \vec{v}_h \right\rangle \\ &\quad = (\vec{f}_c, \vec{v}_h)_{\Omega_c} + \left\langle \hat{\vec{u}}_h \cdot \vec{n}_c - g\hat{\phi}_h + gz, \vec{v}_h \cdot \vec{n}_c \right\rangle - \left\langle \frac{\alpha v\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} P_\tau(\mathbb{K}\nabla\hat{\phi}_h), P_\tau \vec{v}_h \right\rangle, \quad \forall \vec{v}_h \in X_{Sh},\end{aligned} \tag{4.29}$$

$$b_c(\hat{\vec{u}}_h, q_h) = 0, \quad \forall q_h \in Q_{Sh}, \tag{4.30}$$

$$\hat{\phi}_h(t_0) = P_h\phi_0, \quad \hat{\vec{u}}_h(t_0) = \mathbb{P}_h\vec{u}_0. \tag{4.31}$$

5. The first parallel non-iterative domain decomposition method

The basic idea for the first parallel algorithm is to use backward Euler method for the full discretization but approximate ξ_m^{n+1}, ξ_c^{n+1} and $\xi_{c\tau}^{n+1}$ by using the initial conditions at t_0 and numerical solutions at time t_n.

5.1. Algorithm I

In the following, we consider the domain decomposition algorithm for the nth$(n = 0, 1, 2, \ldots, N-1)$ time iteration step.

1. For $n = 0, 1, 2, \ldots, N-1$, by using (3.24)–(3.26), the initial conditions $\hat{\phi}_h^0 = P_h\phi_0$ and $\hat{\vec{u}}_h^0 = \mathbb{P}_h\vec{u}_0$, and the numerical solutions $\hat{\phi}_h^n$ and $\hat{\vec{u}}_h^n$ at t_n, compute

$$\xi_m^n = \hat{\vec{u}}_h^n \cdot \vec{n}_c + g\hat{\phi}_h^n, \tag{5.32}$$

$$\xi_c^n = \hat{\vec{u}}_h^n \cdot \vec{n}_c - g\hat{\phi}_h^n + gz, \tag{5.33}$$

$$\vec{\xi}_{c\tau}^n = \frac{\alpha v\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} P_\tau(\mathbb{K}\nabla\hat{\phi}_h^n). \tag{5.34}$$

2. Independently solve

$$\left(\frac{\hat{\phi}_h^{n+1} - \hat{\phi}_h^n}{\Delta t}, \psi_h\right)_{\Omega_m} + a_m\left(\hat{\phi}_h^{n+1}, \psi_h\right) + \left\langle g\hat{\phi}_h^{n+1}, \psi_h \right\rangle = (f_m^{n+1}, \psi_h)_{\Omega_m} + \left\langle \xi_m^n, \psi_h \right\rangle, \quad \forall \psi_h \in X_{Dh}, \tag{5.35}$$

$$\begin{aligned}&\left(\frac{\hat{\vec{u}}_h^{n+1} - \hat{\vec{u}}_h^n}{\Delta t}, \vec{v}_h\right)_{\Omega_c} + a_c\left(\hat{\vec{u}}_h^{n+1}, \vec{v}_h\right) + b_c(\vec{v}_h, \hat{p}_h^{n+1}) + \left\langle \hat{\vec{u}}_h^{n+1} \cdot \vec{n}_c, \vec{v}_h \cdot \vec{n}_c \right\rangle + \frac{\alpha v\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} \left\langle P_\tau \hat{\vec{u}}_h^{n+1}, P_\tau \vec{v}_h \right\rangle \\ &\quad = \left(\vec{f}_c^{n+1}, \vec{v}_h\right)_{\Omega_c} + \left\langle \xi_c^n, \vec{v}_h \cdot \vec{n}_c \right\rangle - \left\langle \vec{\xi}_{c\tau}^n, P_\tau \vec{v}_h \right\rangle, \quad \forall \vec{v}_h \in X_{Sh},\end{aligned} \tag{5.36}$$

$$b_c(\hat{\vec{u}}_h^{n+1}, q_h) = 0, \quad \forall q_h \in Q_{Sh} \tag{5.37}$$

for $\hat{\phi}_h^{n+1}$, $\hat{\vec{u}}_h^{n+1}$, and $\hat{p}_h^{n+1}$.

5.2. Numerical example

In this subsection, we will use the following numerical example to illustrate the features of the first parallel non-iterative domain decomposition algorithm, including its accuracy and stability. We will use Taylor–Hood element for Stokes equation and quadratic element for the Darcy equation.

Example: Consider the model problem on $\Omega = [0,1] \times [-0.25, 0.75]$ where $\Omega_m = [0,1] \times [0, 0.75]$ and $\Omega_c = [0,1] \times [-0.25, 0]$. $\alpha = 1$, $v = 1$, $g = 1$, $z = 0$, and $\mathbb{K} = kI$ where I the identity matrix and $k = 1$. The boundary condition functions and the source terms are chosen such that the following functions are the exact solutions.

$$\begin{cases} \phi_m = [2 - \pi \sin(\pi x)][-y + cos(\pi(1-y))] \cos(2\pi t), \\ \vec{u}_c = [x^2 y^2 + e^{-y}, -\frac{2}{3} x y^3 + 2 - \pi \sin(\pi x)]^T \cos(2\pi t), \\ p_c = -[2 - \pi \sin(\pi x)] \cos(2\pi y) \cos(2\pi t). \end{cases}$$

All the numerical results below are for t = 1.

We first choose $\Delta t = 8h^3$ and list the errors of the first non-iterative DDM in Table 1. By linear regression, these errors obey

$$\|u_h - u\|_0 \approx 0.9886 h^{3.0370}, \quad |u_h - u|_1 \approx 2.4443 h^{2.1906}, \quad \|p_h - p\|_0 \approx 4.7199 h^{2.4074}, \quad \|\phi_h - \phi\|_0 \approx 2.3667 h^{2.9797},$$
$$|\phi_h - \phi|_1 \approx 4.8451 h^{2.0137}.$$

These results match the regular expectations of accuracy order $O(h^3 + \Delta t)$ arising from backward Euler method, Taylor–Hood element and quadratic element.

Secondly, we choose $\Delta t = h$. The first non-iterative domain decomposition algorithm is still stable, but the errors listed in Table 2 suggests that the accuracy is close to first order. This is expected from the accuracy order $O(h^3 + \Delta t)$.

Thirdly, we choose $\Delta t = \sqrt{h}$ and $\Delta t = 4h$ to further investigate the stability of Algorithm I. The numerical results in Tables 3 and 4 are still stable even though the accuracy decreases due to larger Δt. This is an interesting observation since the numerical results from the previous time iteration step is used to predict the parameters for the current step in an explicit way. And the corresponding stability analysis is an interesting future work.

Table 1
Errors of the first non-iterative DDM for $\Delta t = 8h^3$.

h	$\|u_h - u\|_0$	$\|u_h - u\|_1$	$\|p_h - p\|_0$	$\|\phi_h - \phi\|_0$	$\|\phi_h - \phi\|_1$
1/8	1.8244×10^{-3}	2.7194×10^{-2}	3.4486×10^{-2}	4.7632×10^{-3}	7.3861×10^{-2}
1/12	5.1366×10^{-4}	1.0154×10^{-2}	1.1281×10^{-2}	1.4546×10^{-3}	3.2439×10^{-2}
1/16	2.1483×10^{-4}	5.3990×10^{-3}	5.5690×10^{-3}	6.1663×10^{-4}	1.8165×10^{-2}
1/20	1.0982×10^{-4}	3.3672×10^{-3}	3.3284×10^{-3}	3.1585×10^{-4}	1.1603×10^{-2}
1/24	6.3561×10^{-5}	2.3058×10^{-3}	2.2205×10^{-3}	1.8269×10^{-4}	8.0496×10^{-3}
1/28	4.0049×10^{-5}	1.6799×10^{-3}	1.5906×10^{-3}	1.1496×10^{-4}	5.9109×10^{-3}
1/32	2.6846×10^{-5}	1.2792×10^{-3}	1.1973×10^{-3}	7.6963×10^{-5}	4.5242×10^{-3}

Table 2
Errors of the first non-iterative DDM for $\Delta t = h$.

h	$\|u_h - u\|_0$	$\|u_h - u\|_1$	$\|p_h - p\|_0$	$\|\phi_h - \phi\|_0$	$\|\phi_h - \phi\|_1$
1/8	2.8310×10^{-2}	4.0257×10^{-1}	4.9463×10^{-1}	2.5244×10^{-2}	1.4670×10^{-1}
1/16	1.0396×10^{-2}	1.3079×10^{-1}	1.4506×10^{-1}	1.5072×10^{-2}	7.5069×10^{-2}
1/32	4.2258×10^{-3}	4.7188×10^{-2}	5.3956×10^{-2}	8.4399×10^{-3}	3.9873×10^{-2}
1/64	1.8882×10^{-3}	1.9391×10^{-2}	2.3967×10^{-2}	4.4860×10^{-3}	2.0802×10^{-2}

Table 3
Errors of the first non-iterative DDM for $\Delta t = \sqrt{h}$.

h	$\|u_h - u\|_0$	$\|u_h - u\|_1$	$\|p_h - p\|_0$	$\|\phi_h - \phi\|_0$	$\|\phi_h - \phi\|_1$
1/16	3.8980×10^{-2}	1.3399×10^{0}	1.4279×10^{0}	3.8980×10^{-2}	3.8980×10^{-1}
1/36	4.8068×10^{-2}	7.1271×10^{-1}	6.7836×10^{-1}	3.0056×10^{-2}	1.5986×10^{-1}
1/64	3.0636×10^{-2}	4.3674×10^{-1}	4.0150×10^{-1}	2.4734×10^{-2}	1.2801×10^{-1}
1/100	2.1632×10^{-2}	2.9799×10^{-1}	2.7240×10^{-1}	2.1179×10^{-2}	1.0706×10^{-1}

Table 4
Errors of the first non-iterative DDM for $\Delta t = 4h$.

h	$\|u_h - u\|_0$	$\|u_h - u\|_1$	$\|p_h - p\|_0$	$\|\phi_h - \phi\|_0$	$\|\phi_h - \phi\|_1$
1/16	3.8980×10^{-2}	1.3399×10^{0}	1.4279×10^{0}	3.8980×10^{-2}	3.8980×10^{-1}
1/32	3.0358×10^{-2}	4.3176×10^{-1}	4.1615×10^{-1}	2.4786×10^{-2}	1.2818×10^{-1}
1/64	1.0695×10^{-2}	1.3534×10^{-1}	1.3273×10^{-1}	1.4975×10^{-2}	7.2658×10^{-2}
1/128	4.2719×10^{-3}	4.7940×10^{-2}	5.2260×10^{-2}	8.4211×10^{-3}	3.9562×10^{-2}

6. The second non-iterative domain decomposition method

The numerical results in Section 5.2 imply that the accuracy of the first non-iterative domain decomposition method is $O(h^3+\Delta t)$ in L^2 norm for $\hat{\vec{u}}_c$ and $\hat{\phi}_m$. Hence, in order to obtain the third order accuracy, Δt needs to be proportional to h^3. However, in practice a larger Δt is usually needed in order to dramatically reduce the computation cost in time iteration and make the numerical simulation efficient. To overcome this shortcoming of the first non-iterative DDM, in this section we propose the second non-iterative DDM, which employs the three-step backward differentiation and the idea of predictor–corrector. The new method improves the accuracy order to be $O(h^3+\Delta t^3)$ so that third order accuracy can be still achieved when $\Delta t = h$.

6.1. Algorithm II

In the following, We consider the domain decomposition algorithm for the nth$(n=0,1,2,\ldots,N-1)$ time iteration step.

1. For $n=0$, by using the initial conditions $\hat{\phi}_h^0 = P_h\phi_0$ and $\hat{\vec{u}}_h^0 = \mathbb{P}_h\vec{u}_0$, and (3.24)–(3.26), compute $\xi_m^0 = \hat{\vec{u}}_h^0\cdot\vec{n}_c + g\hat{\phi}_h^0$, $\xi_c^0 = \hat{\vec{u}}_h^0\cdot\vec{n}_c - g\hat{\phi}_h^0 + gz$, and $\vec{\xi}_{c\tau}^0 = \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}P_\tau(\mathbb{K}\nabla\hat{\phi}_h^0)$.
2. Using backward Euler method, independently solve

$$\left(\frac{\hat{\phi}_h^1-\hat{\phi}_h^0}{\Delta t},\psi_h\right)_{\Omega_m} + a_m\left(\hat{\phi}_h^1,\hat{\phi}_h\right) + \left\langle g\hat{\phi}_h^1,\psi_h\right\rangle = (f_m^1,\psi_h)_{\Omega_m} + \langle\xi_m^0,\psi_h\rangle,\quad \forall\psi_h\in X_{Dh},\left(\frac{\hat{\vec{u}}_h^1-\hat{\vec{u}}_h^0}{\Delta t},\vec{v}_h\right)_{\Omega_c}$$
$$+ a_c(\hat{\vec{u}}_h^1,\vec{v}_h) + b_c(\vec{v}_h,\hat{p}_h^1) + \left\langle\hat{\vec{u}}_h^1\cdot\vec{n}_c,\vec{v}_h\cdot\vec{n}_c\right\rangle + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\left\langle P_\tau\hat{\vec{u}}_h^1,P_\tau\vec{v}_h\right\rangle$$
$$= (\vec{f}_c^1,\vec{v}_h)_{\Omega_c} + \langle\xi_c^0,\vec{v}_h\cdot\vec{n}_c\rangle - \left\langle\vec{\xi}_{c\tau}^0,P_\tau\vec{v}_h\right\rangle,\forall\vec{v}_h\in X_{Sh}, b_c(\hat{\vec{u}}_h^1,q_h)=0,\quad\forall q_h\in Q_{Sh}$$

   for $\hat{\phi}_h^1$, $\hat{\vec{u}}_h^1$, and $\hat{p}_h^1$.
3. Compute $\xi_m^1 = \hat{\vec{u}}_h^1\cdot\vec{n}_c + g\hat{\phi}_h^1$, $\xi_c^1 = \hat{\vec{u}}_h^1\cdot\vec{n}_c - g\hat{\phi}_h^1 + gz$, and $\vec{\xi}_{c\tau}^1 = \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}P_\tau(\mathbb{K}\nabla\hat{\phi}_h^1)$.
4. Using Crank–Nicolson, independently solve

$$\left(\frac{\hat{\phi}_h^1-\hat{\phi}_h^0}{\Delta t},\psi_h\right)_{\Omega_m} + a_m\left(\frac{\hat{\phi}_h^1+\hat{\phi}_h^0}{2},\psi_h\right) + \left\langle g\frac{\hat{\phi}_h^1+\hat{\phi}_h^0}{2},\psi_h\right\rangle$$
$$= \left(\frac{f_m^1+f_m^0}{2},\psi_h\right)_{\Omega_m} + \left\langle\frac{\xi_m^1+\xi_m^0}{2},\psi_h\right\rangle,\forall\psi_h\in X_{Dh},\left(\frac{\hat{\vec{u}}_h^1-\hat{\vec{u}}_h^0}{\Delta t},\vec{v}_h\right)_{\Omega_c} + a_c\left(\frac{\hat{\vec{u}}_h^1+\hat{\vec{u}}_h^0}{2},\vec{v}_h\right) + b_c\left(\vec{v}_h,\frac{\hat{p}_h^1+\hat{p}_h^0}{2}\right)$$
$$+ \left\langle\frac{\hat{\vec{u}}_h^1+\hat{\vec{u}}_h^0}{2}\cdot\vec{n}_c,\vec{v}_h\cdot\vec{n}_c\right\rangle + \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}\left\langle P_\tau\frac{\hat{\vec{u}}_h^1+\hat{\vec{u}}_h^0}{2},P_\tau\vec{v}_h\right\rangle = \left(\frac{\vec{f}_c^1+\vec{f}_c^0}{2},\vec{v}_h\right)_{\Omega_c} + \left\langle\frac{\xi_c^1+\xi_c^0}{2},\vec{v}_h\cdot\vec{n}_c\right\rangle$$
$$- \left\langle\frac{\vec{\xi}_{c\tau}^1+\vec{\xi}_{c\tau}^0}{2},P_\tau\vec{v}_h\right\rangle,\quad\forall\vec{v}_h\in X_{Sh},$$

$$b_c\left(\frac{\hat{\vec{u}}_h^1+\hat{\vec{u}}_h^0}{2},q_h\right)=0,\quad\forall q_h\in Q_{Sh},$$

   to update $\hat{\phi}_h^1$, $\hat{\vec{u}}_h^1$, and $\hat{p}_h^1$.
5. Update

$$\xi_m^1 = \hat{\vec{u}}_h^1\cdot\vec{n}_c + g\hat{\phi}_h^1, \tag{6.38}$$
$$\xi_c^1 = \hat{\vec{u}}_h^1\cdot\vec{n}_c - g\hat{\phi}_h^1 + gz, \tag{6.39}$$
$$\vec{\xi}_{c\tau}^1 = \frac{\alpha\nu\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}}P_\tau(\mathbb{K}\nabla\hat{\phi}_h^1). \tag{6.40}$$

6. For $n=1$, approximate $\xi_m^2\approx 2\xi_m^1-\xi_m^0$, $\xi_c^2\approx 2\xi_c^1-\xi_c^0$, and $\vec{\xi}_{c\tau}^2\approx 2\vec{\xi}_{c\tau}^1-\vec{\xi}_{c\tau}^0$.
7. Using two-step backward differentiation, independently solve

$$\left(\frac{3\hat{\phi}_h^2-4\hat{\phi}_h^1+\hat{\phi}_h^0}{2\Delta t},\psi_h\right)_{\Omega_m} + a_m\left(\hat{\phi}_h^2,\psi_h\right) + \left\langle g\hat{\phi}_h^2,\psi_h\right\rangle = (f_m^2,\psi_h)_{\Omega_m} + \langle\xi_m^2,\psi_h\rangle,\quad\forall\psi_h\in X_{Dh},$$

$$\left(\frac{3\hat{\vec{u}}_h^2 - 4\hat{\vec{u}}_h^1 + \hat{\vec{u}}_h^0}{2\Delta t}, \vec{v}_h\right)_{\Omega_c} + a_c(\hat{\vec{u}}_h^2, \vec{v}_h) + b_c(\vec{v}_h, \hat{p}_h^2) + \left\langle \hat{\vec{u}}_h^2 \cdot \vec{n}_c, \vec{v}_h \cdot \vec{n}_c \right\rangle + \frac{\alpha v\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} \left\langle P_\tau(\hat{\vec{u}}_h^2), P_\tau \vec{v}_h \right\rangle$$
$$= \left(\vec{f}_c^2, \vec{v}_h\right)_{\Omega_c} + \langle \xi_c^2, \vec{v}_h \cdot \vec{n}_c \rangle - \left\langle \vec{\xi}_{c\tau}^2, P_\tau \vec{v}_h \right\rangle, \quad \forall \vec{v}_h \in X_{Sh},$$

$$b_c\left(\hat{\vec{u}}_h^2, q_h\right) = 0, \quad \forall q_h \in Q_{Sh}$$

for $\hat{\phi}_h^2, \hat{\vec{u}}_h^2$, and $\hat{p}_h^2$.

8. Update

$$\xi_m^2 = \hat{\vec{u}}_h^2 \cdot \vec{n}_c + g\hat{\phi}_h^2, \tag{6.41}$$
$$\xi_c^2 = \hat{\vec{u}}_h^2 \cdot \vec{n}_c - g\hat{\phi}_h^2 + gz, \tag{6.42}$$
$$\vec{\xi}_{c\tau}^2 = \frac{\alpha v\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} P_\tau\left(\mathbb{K}\nabla\hat{\phi}_h^2\right). \tag{6.43}$$

9. For $n = 2, \ldots, N-1$, by using the numerical solution at t_n and (3.24)–(3.26), approximate $\xi_m^{n+1} \approx 3\xi_m^n - 3\xi_m^{n-1} + \xi_m^{n-2}$, $\xi_c^{n+1} \approx 3\xi_c^n - 3\xi_c^{n-1} + \xi_c^{n-2}$, and $\vec{\xi}_{c\tau}^{n+1} \approx 3\vec{\xi}_{c\tau}^n - 3\vec{\xi}_{c\tau}^{n-1} + \vec{\xi}_{c\tau}^{n-2}$.
10. Using three-step backward differentiation, independently solve

$$\left(\frac{11\hat{\phi}_h^{n+1} - 18\hat{\phi}_h^n + 9\phi_h^{n-1} - 2\hat{\phi}_h^{n-2}}{6\Delta t}, \psi_h\right)_{\Omega_m} + a_m\left(\hat{\phi}_h^{n+1}, \psi_h\right) + \left\langle g\hat{\phi}_h^{n+1}, \psi_h \right\rangle = \left(f_m^{n+1}, \psi_h\right)_{\Omega_m} + \langle \xi_m^{n+1}, \psi_h \rangle, \forall \psi_h \in X_{Dh}, \tag{6.44}$$

$$\left(\frac{11\hat{\vec{u}}_h^{n+1} - 18\hat{\vec{u}}_h^n + 9\hat{\vec{u}}_h^{n-1} - 2\hat{\vec{u}}_h^{n-2}}{6\Delta t}, \vec{v}_h\right)_{\Omega_c} + a_c\left(\hat{\vec{u}}_h^{n+1}, \vec{v}_h\right) + b_c(\vec{v}_h, \hat{p}_h^{n+1}) + \left\langle \hat{\vec{u}}_h^{n+1} \cdot \vec{n}_c, \vec{v}_h \cdot \vec{n}_c \right\rangle$$
$$+ \frac{\alpha v\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} \left\langle P_\tau \hat{\vec{u}}_h^{n+1}, P_\tau \vec{v}_h \right\rangle = \left(\vec{f}_c^{n+1}, \vec{v}_h\right)_{\Omega_c} + \langle \xi_c^{n+1}, \vec{v}_h \cdot \vec{n}_c \rangle - \left\langle \vec{\xi}_{c\tau}^{n+1} P_\tau \vec{v}_h \right\rangle, \forall \vec{v}_h \in X_{Sh}, \tag{6.45}$$

$$b_c\left(\hat{\vec{u}}_h^{n+1}, q_h\right) = 0, \quad \forall q_h \in Q_{Sh} \tag{6.46}$$

for $\hat{\phi}_h^{n+1}$, $\hat{\vec{u}}_h^{n+1}$, and $\hat{p}_h^{n+1}$.

11. Update

$$\xi_m^{n+1} = \hat{\vec{u}}_h^{n+1} \cdot \vec{n}_c + g\hat{\phi}_h^{n+1}, \tag{6.47}$$
$$\xi_c^{n+1} = \hat{\vec{u}}_h^{n+1} \cdot \vec{n}_c - g\hat{\phi}_h^{n+1} + gz, \tag{6.48}$$
$$\vec{\xi}_{c\tau}^{n+1} = \frac{\alpha v\sqrt{\mathbf{d}}}{\sqrt{\mathrm{trace}(\prod)}} P_\tau\left(\mathbb{K}\nabla\hat{\phi}_h^{n+1}\right). \tag{6.49}$$

Remark 6.1. The three-step backward differentiation needs the solutions at the first three time iteration steps ($n = 1, 2, 3$) to start the iteration. Because only the initial condition at $n = 0$ is given, the solutions at $n = 1$ and $n = 2$ need to be approximated numerically. Since the global error of three-step backward differentiation [6], the local truncation error of two-step backward differentiation [55], and the local truncation error of Crank–Nicolson scheme [66] are all of order $O(\Delta t^3)$, the numerical solutions of the second parallel non-iterative domain decomposition method at $n = 1$ and $n = 2$ do not deteriorate the global accuracy (see, e.g., Section 10.3 in [55]). Hence one can expect the global accuracy order of the second parallel non-iterative domain decomposition method is $O(h^3 + \Delta t^3)$ if Taylor–Hood element is used for Stokes equation and quadratic element is used for the Darcy equation. This will be numerically verified in the following subsection.

6.2. Numerical example

To illustrate the numerical behavior of the second non-iterative DDM (Algorithm II), we consider the same example in the previous section and also choose $\Delta t = h$. From the data and the linear regression below, we can see that the second algorithm is stable and the accuracy order is $O(h^3 + \Delta t^3)$. Table 5 contains the errors of the second non-iterative domain decomposition algorithm. By linear regression, the errors in Table 5 obey

$$\|u_h - u\|_0 \approx 3.7036h^{3.7036}, \quad |u_h - u|_1 \approx 522.45h^{3.3198}, \quad \|p_h - p\|_0 \approx 652.09h^{3.3863}, \quad \|\phi_h - \phi\|_0 \approx 1.8714h^{2.7746},$$
$$|\phi_h - \phi|_1 \approx 5.2780h^{2.0302}.$$

We also choose $\Delta t = \sqrt{h}$ and $\Delta t = 4h$ to further investigate the stability of Algorithm II. The corresponding numerical results are still stable even though they are less accurate than those of Table 5 because of larger Δt. This performance is similar to that of Tables 3 and 4. Hence we omit the corresponding datum here to reduce the length of the presentation.

Table 5
Errors of the second non-iterative DDM for $\Delta t = h$.

h	$\|u_h - u\|_0$	$\|u_h - u\|_1$	$\|p_h - p\|_0$	$\|\phi_h - \phi\|_0$	$\|\phi_h - \phi\|_1$
1/8	4.7982×10^{-2}	7.4699×10^{-1}	7.8154×10^{-1}	6.2885×10^{-3}	7.9476×10^{-2}
1/16	2.9626×10^{-3}	4.6352×10^{-2}	5.1155×10^{-2}	7.2116×10^{-4}	1.8563×10^{-2}
1/32	2.2817×10^{-4}	3.6045×10^{-3}	3.5519×10^{-3}	1.3457×10^{-4}	4.5593×10^{-3}
1/64	1.8125×10^{-5}	3.9907×10^{-4}	3.7080×10^{-4}	1.9298×10^{-5}	1.1335×10^{-3}
1/128	1.6344×10^{-6}	8.1105×10^{-5}	7.3452×10^{-5}	2.5619×10^{-6}	2.8291×10^{-3}

7. Conclusions

In this article we propose two parallel non-iterative multi-physics domain decomposition methods for non-stationary Stoke–Darcy model with Beavers–Joseph interface condition. Both of the two methods possess the optimal convergence rates expected from the finite elements and time discretization schemes used.

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