Nyström Methods and Extrapolation for Solving Steklov Eigensolutions and Its Application in Elasticity

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Based on potential theory, Steklov eigensolutions of elastic problems can be converted into eigenvalue problems of boundary integral equations (BIEs). The kernels of these BIEs are characterized by logarithmic and Hilbert singularities. In this article, the Nyström methods are presented for obtaining eigensolutions $(\lambda^{(i)}, u^{(i)})$, which have to deal with the two kinds of singularities simultaneously. The solutions possess high accuracy orders $O(h^3)$ and an asymptotic error expansion with odd powers. Using h^3-Richardson extrapolation algorithms, we can greatly improve the accuracy orders to $O(h^5)$. Furthermore, a generalized Fourier series is constructed by the eigensolutions, and then solving the elasticity displacement and traction problems involves just calculating the coefficients of the series. A class of elasticity problems with boundary Γ is solved with high convergence rate $O(h^5)$. The efficiency is illustrated by a numerical example. Numer Methods Partial Differential Eq 28: 2021–2040, 2012

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I. INTRODUCTION

The fundamental boundary eigenproblems for planar elastostatics are defined as follows: to find a non-zero deformation $u = (u_1, u_2)$ in the domain Ω and on the boundary Γ satisfying

$$\begin{cases} \sigma_{ij,j} = C_{ijkl}u_{k,lj} = 0, \text{ in } \Omega, \\ t_i = \lambda u_i, \text{ on } \Gamma, \ k,l,i,j = 1,2, \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded, simply connected domain with a smooth boundary Γ, t_i is a traction vector on Γ, $\sigma_{ij} = C_{ijkl}u_{k,l}$ is a stress tensor, C_{ijkl} is an elastic constant of the tensor, and λ is an eigenvalue. Following vector computational rules, the repeated subscripts imply the summation from 1 to 2.

The problems are called Steklov eigenproblems [1, 2] and the advantages compared with traditional eigenproblems are as follows:

1. the eigenfunctions $\tilde{u}$ associated with Eq.(1) satisfy the governing equation in the domain Ω;
2. the infinite sequence of eigenfunctions $\{\tilde{u}_{(l)}\}$ can be used as basic functions for all solutions in domain Ω governed by Eq.(1) with arbitrary well-defined conditions on Γ.

The governing equations in Eq.(1) have been widely applied in many physical problems, such as for a cantilever beam, for a simply supported beam, for a plate with edge notch, edge crack, bimaterial, circular holes, and so on [2, 3].

To obtain eigensolutions $\lambda^{(l)}$ and $u^{(l)}$, Eq.(1) is converted into the following boundary integral equations [4–7] (BIEs) by potential theory:

$$\alpha_{ij}(y)u_j^{(l)}(y) + \int_\Gamma k_{ij}^*(y,x)u_j^{(l)}(x)ds_x = \lambda^{(l)} \int_\Gamma h_{ij}^*(y,x)u_j^{(l)}(x)ds_x, \tag{2}$$

where $\alpha_{ij}(y)$ is related to the interior angle of tangent lines at $y \in \Gamma$, in particular, when y is not a singular point, then $\alpha_{ij} = \delta_{ij}/2$, and

$$\begin{cases} h_{ij}^* = \dfrac{1}{8\pi\mu(1-\nu)}[-(3-4\nu)\delta_{ij}\ln r + r_{,i}r_{,j}], \\ k_{ij}^* = \dfrac{1}{4\pi(1-\nu)r}\left[\dfrac{\partial r}{\partial n}((1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}) + (1-2\nu)(n_i r_{,j} - n_j r_{,i})\right], \end{cases}$$

are Kelvin's fundamental solutions [2, 8], where μ is the shear modulus, ν is the Poisson ratio, $r = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$ is the distance between x and y, $r_{,i} = \partial r/\partial x_i$, and $n = (n_1, n_2)$ is the unit outward normal on Γ. Equation (2) are obviously singular integral equations. In particular, the second term of the left-hand side is characterized as a Hilbert singularity and the term of the right-hand side is characterized as a logarithmic singularity. Consequently, the key to finding the eigensolutions $\{\lambda^{(l)}, u^{(l)}\}$ accurately is converted to dealing with the logarithmic and Hilbert singularities, respectively.

A considerable number of articles have researched the relevant problems. Alves and Antunes [9] studied the application of the fundamental solutions method for solving the eigenvalue problem for the biharmonic operator. Constanda [10] solved the interior and exterior Dirichlet and

Neumann problems of plane elasticity by real variables BIE method. Müller and Heise [11] calculated the eigenvalues for plane elastostatic boundary value problems to construct the condition numbers and to estimate the truncation error. Cheng et al., [12, 13] discussed Steklov eigenvalue and its extrapolation for Laplace equation with smooth or polygonal boundary. Auchmuty [14] described some properties and applications of Steklov eigenproblems for prototypical second-order elliptic operators on bounded regions. Del Pezzo et al. [15] introduced the first Steklov eigenvalue in a bounded smooth domain, and analyzed the dependence of the first eigenvalue on some parameters. Parton and Perlin [16] introduced the eigenvalue λ into the boundary conditions of the elasticity problem and got some analytical solutions in a circular isotropic elastic body. Hadjesfandiari and Dargush [1, 2, 17] gave the general theory of fundamental boundary eigensolutions for elasticity and potential problems. They showed the theorems of the generalized discrete Fourier series constructed by the eigenvalues and the eigenvectors. They also used the finite element method to achieve the error estimate of the approximate solution. Talbot and Crampton [18] approached 2D vibrational problems by a pseudospectral method since the governing partial differential equations were translated into a matrix eigenvalue problem, which was solved by a collocation method. Torńe [19] outlined a method of obtaining a sequence of eigenvalues using infinite dimensional Ljusternik–Schnirelman theory and investigated some nodal properties of eigenfunctions associated with the first and second eigenvalues.

We have concisely expounded [20] the Nyström methods for solving the Steklov eigenproblems and some numerical examples are shown for determination of eigenvalues. Compared with the article [20], theoretical analysis is thoroughly carried out to obtain an asymptotic error expansion with odd powers for the approximate solutions in this article. Then extrapolation algorithms (EAs) [21–24] are established and the convergence rate is $O(h^5)$. On the other hand, based on the generalized Fourier series, the Steklov eigensolutions are applied to Dirichlet and Neumann boundary condition for elasticity by just calculating the coefficients of the series and the convergence rate $O(h^5)$ is obtained. Additionally, we derive an a posteriori error estimate for constructing adaptive algorithms. A numerical example verifies the theoretic results and illustrates the features of Nyström methods in practice.

This article is organized as follows: in section 2, we construct the Nyström methods and obtained an asymptotic error expansion theoretically; in section 3, we construct the EAs and obtain an a posteriori error estimate; in section 4, a generalized Fourier series is used for elasticity problems; in section 5, a numerical example shows the significance of the algorithms.

II. NYSTRÖM METHODS

Define boundary integral operators on Γ as follows:

$$\begin{cases} (K_{ij}w)(y) = \displaystyle\int_\Gamma k^*_{ij}(y,x)w(x)ds_x \qquad y \in \Gamma, i,j = 1,2, \\ (H_{ij}w)(y) = \displaystyle\int_\Gamma h^*_{ij}(y,x)w(x)ds_x \qquad y \in \Gamma, i,j = 1,2. \end{cases} \tag{3}$$

Then Eq. (2) can be converted into the following operator equations:

$$\begin{pmatrix} \frac{1}{2}I_0 + K_{11} & K_{12} \\ K_{21} & \frac{1}{2}I_0 + K_{22} \end{pmatrix} \begin{pmatrix} u_1^{(l)} \\ u_2^{(l)} \end{pmatrix} = \lambda^{(l)} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} u_1^{(l)} \\ u_2^{(l)} \end{pmatrix}, \tag{4}$$

where I_0 is an identity operator.

Assume that Γ can be described by a regular parametric mapping $x(s) = (x_1(s), x_2(s)) : (0, 2\pi] \to \Gamma$, satisfying $|x'(s)|^2 = |x_1'(s)|^2 + |x_2'(s)|^2 > 0$, and $x_i(s) \in C^{2m+1}[0, 2\pi], i = 1, 2$. Define the integral operator on $C^{2m+1}[0, 2\pi]$:

$$
\begin{aligned}
(A_0\omega)(t) &= \int_0^{2\pi} a_0(t,\tau)\omega(\tau)|x'(\tau)|d\tau \\
&= \bar{c}_0 \int_0^{2\pi} \ln\left|2e^{-1/2}\sin\left(\frac{t-\tau}{2}\right)\right| \omega(\tau)|x'(\tau)|d\tau, \\
(B_0\omega)(t) &= \int_0^{2\pi} b_0(t,\tau)\omega(\tau)|x'(\tau)|d\tau \\
&= \bar{c}_0 \int_0^{2\pi} \ln\left|\frac{x(t)-x(\tau)}{2e^{-1/2}\sin((t-\tau)/2)}\right| \omega(\tau)|x'(\tau)|d\tau, \\
(B_{ij}\omega)(t) &= \int_0^{2\pi} b_{ij}(t,\tau)\omega(\tau)|x'(\tau)|d\tau \\
&= c_1 \int_0^{2\pi} \frac{(x_i(t)-x_i(\tau))(x_j(t)-x_j(\tau))}{|x(t)-x(\tau)|^2}\omega(\tau)|x'(\tau)|d\tau, \\
(C_0\omega)(t) &= \int_0^{2\pi} c_0(t,\tau)\omega(\tau)|x'(\tau)|d\tau \\
&= c_2 \int_0^{2\pi} \{(n_1 r_{.2} - n_2 r_{.1})/r\}\omega(\tau)|x'(\tau)|d\tau, \\
(M_{ii}\omega)(t) &= \int_0^{2\pi} m_{ii}(t,\tau)\omega(\tau)|x'(\tau)|d\tau \\
&= c_3 \int_0^{2\pi} \left\{\left[\frac{\partial r}{\partial n}[(1-2\upsilon)+2r_{.i}r_{.i}]/r\right\} \omega(\tau)|x'(\tau)|d\tau, \\
(M_{ij}\omega)(t) &= \int_0^{2\pi} m_{ij}(t,\tau)\omega(\tau)|x'(\tau)|d\tau \\
&= c_3 \int_0^{2\pi} \left\{\frac{\partial r}{\partial n}(2r_{.i}r_{.j})/r\right\} \omega(\tau)|x'(\tau)|d\tau \quad i \neq j,
\end{aligned}
$$

where $\bar{c}_0 = -(3-4\nu)/[8\pi\mu(1-\nu)]$, $c_1 = 1/[8\pi\mu(1-\nu)]$, $c_2 = -(1-2\nu)/[4\pi(1-\nu)]$, $c_3 = -1/[4\pi(1-\nu)]$. As $t \to s$, depending on the properties of the kernels and using a Taylor expansion, we know that A_0 is a logarithmic weak singular operator [4], and B_0, B_{ij}, and M_{ij} are smooth operators. C_0 is a Hilbert singularity operator as its kernel contains

$$\frac{n_i r_{.j} - n_j r_{.i}}{r} = (-1)^i \frac{1 + O(t-s)}{(t-s) + O(t-s)} \quad i \neq j. \tag{5}$$

Then Eq.(4) is equivalent to

$$
\begin{cases}
\left(\dfrac{1}{2}I + C + M\right)u^{(l)} = \lambda^{(l)}(A+B)u^{(l)}, \\
\|u^{(l)}\|_{0,\Gamma}^2 = \displaystyle\int_0^{2\pi} |u^{(l)}(s)|^2|x'(s)|ds = 1,
\end{cases} \tag{6}
$$

where

$$I = \begin{pmatrix} I_0 & 0 \\ 0 & I_0 \end{pmatrix}, \quad A = \begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix}, \quad B = \begin{pmatrix} B_0 + B_{11} & B_{12} \\ B_{21} & B_0 + B_{22} \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & C_0 \\ -C_0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

A. Nyström Approximation

Let $h = \pi/n$, $(n \in N)$ be the mesh width and $t_j = jh$, $(j = 0, 1, \cdots, 2n-1)$ be the nodes. As B_0, B_{ij}, and M_{ij} are smooth integral operators with the period 2π, we can obtain the highly accurate Nyström approximation by the midpoint rule [12, 13]. For example, the Nyström approximation operator B_0^h of B_0 can be defined as:

$$\left(B_0^h \omega\right)(t) = h \sum_{j=0}^{2n-1} b_0(t, \tau_j)\omega(\tau_j), \tag{7}$$

and the error is

$$(B_0\omega)(t) - \left(B_0^h \omega\right)(t) = O(h^{2m}). \tag{8}$$

The Nyström approximation B_{ij}^h of B_{ij} and M_{ij}^h of M_{ij} can be defined similarly.

For the logarithmic singular operator A_0, the continuous approximation of its kernel $a_n(t, \tau)$ is defined as:

$$a_n(t, \tau) = \begin{cases} a_0(t, \tau), \ for \ |t - \tau| \geq h, \\ \bar{c}_0 h \ln |e^{-1/2} h/(2\pi)|, \ for \ |t - \tau| < h, \end{cases} \tag{9}$$

and by Sidi's quadrature rules [24], its Nyström approximation operator A_0^h can be defined as:

$$\left(A_0^h \omega\right)(t) = h \sum_{j=0}^{2n-1} a_n(t, \tau_j)\omega(\tau_j)|x'(\tau_j)|, \tag{10}$$

which has the following error estimate

$$(A_0\omega)(t) - \left(A_0^h \omega\right)(t) = 2 \sum_{\mu=1}^{m-1} \frac{\varsigma'(-2\mu)}{(2\mu)!} \omega^{(2\mu)}(t) h^{2\mu+1} + O(h^{2m}), \tag{11}$$

where $\varsigma'(t)$ is the derivative of Riemann zeta function.

Because C_0 is a Hilbert singular operator, its Nyström approximation operator C_0^h can be defined by Sidi's quadrature rules [24]:

$$\left(C_0^h \omega\right)(t_i) = 2c_2 a_1(t_i, t_i) h \sum_{j=0}^{2n-1} \cot((t_j - t_i)/2)\omega(t_j)|x'(t_j)|\varepsilon_{ij}, \tag{12}$$

where $t_i = ih$, $i = 0, \cdots, 2n-1$, $h = \pi/n$,

$$a_1(t, s) = \frac{1}{(t - s) + O(t - s)} \frac{\tan((t - s)/2)}{1/2}, \tag{13}$$

and

$$\varepsilon_{ij} = \begin{cases} 1, & \text{if } |i-j| \text{ is odd number} \\ 0, & \text{if } |i-j| \text{ is even number} \end{cases}. \tag{14}$$

The Nyström approximation has the following error bounds

$$(C_0\omega)(t_i) - (C_0^h\omega)(t_i) = O(h^{2m}). \tag{15}$$

Thus, we obtain the numerical approximate of Eq. (6),

$$\begin{cases} \left(\frac{1}{2}I + C^h + M^h\right) u_h^{(l)} = \lambda_h^{(l)}(A^h + B^h)u_h^{(l)}, \\ h\sum_{i=1}^{2}\sum_{j=0}^{2n-1}\left(u_{ih}^{(l)}(t_j)\right)^2|x'(t_j)| = 1, \end{cases} \tag{16}$$

where A^h, B^h, C^h, and M^h are discrete matrices of order $4n$ corresponding to the operators A, B, C, and M, respectively. $\lambda_h^{(l)}$ and $u_h^{(l)}$ are the approximate solution of eigenvalue $\lambda^{(l)}$ and eigenvector $u^{(l)}$, respectively.

B. Asymptotically Compact Convergence

Define $D_1^h = \text{diag}(a_1(t_0,t_0),\cdots,a_1(t_{2n-1},t_{2n-1}))$, and C_1^h is a circulant matrix:

$$C_1^h = 2h \text{ circulant } \left(0, -\cot\left(\frac{\pi}{2n}\right), 0, \cdots, 0, -\cot\frac{(2n-1)\pi}{2n}\right).$$

Let

$$C_0^h = c_2 D_1^h C_1^h, C_2^h = \begin{pmatrix} 0 & C_1^h \\ -C_1^h & 0 \end{pmatrix}, D^h = \text{diag}(D_1^h, D_1^h), \tag{17}$$

we have

$$C^h = c_2 D^h C_2^h = \begin{pmatrix} 0 & C_0^h \\ -C_0^h & 0 \end{pmatrix}. \tag{18}$$

Lemma 1. *The eigenvalues of the discrete matrix C_1^h consist of*

$$\rho_k = \begin{cases} 0, & \texttt{if } k = 0, n; \\ 2\pi i, & \texttt{if } 1 \le k \le n-1; \\ -2\pi i, & \texttt{if } n+1 \le k \le 2n-1; i = \sqrt{-1}. \end{cases} \tag{19}$$

Proof. By properties of antisymmetric circulant matrix, the eigenvalue of C_1^h must be an imaginary number and $\rho_k = f(\xi_k)$ with $f(z) = \sum_{j=1}^{n} z^{2j-1}\cot(\frac{(2j-1)\pi}{2n})$ and $\xi_k = \exp(2\pi ki/(2n))$, $k = 0, 1, \ldots, 2n-1$. We can obtain

$$\rho_k = \sum_{j=1}^{n}\frac{2h}{2\pi}z_k^{2j-1}\cot\left(\frac{(2j-1)\pi}{2n}\right) = \frac{i}{n}\sum_{j=1}^{n}\sin\frac{2\pi k(2j-1)}{2n}\cot\frac{(2j-1)\pi}{2n},$$

where $z_k = \exp(2\pi ki/2n)$, $k = 0, \dots, 2n-1$. Obviously $\rho_k = -\rho_{2n-k}$, $1 \le k \le n$, and $\rho_0 = \rho_n = 0$. We have

$$\rho_1 = \frac{i}{n}\sum_{j=1}^{n} \sin\frac{2\pi(2j-1)}{2n}\cot\frac{(2j-1)\pi}{2n} = \frac{2i}{n}\sum_{j=1}^{n}\cos^2\frac{(2j-1)\pi}{2n} = 2\pi i,$$

and

$$\rho_2 = \frac{i}{n}\sum_{j=1}^{n} \sin\frac{4\pi(2j-1)}{2n}\cot\frac{(2j-1)\pi}{2n} = \frac{4i}{n}\sum_{j=1}^{n}\cos\frac{2\pi(2j-1)}{2n}\cos^2\frac{(2j-1)\pi}{2n}$$
$$= \frac{2i}{n}\sum_{j=1}^{n}\cos\frac{2\pi(2j-1)}{2n}\left[1+\cos\frac{2\pi(2j-1)}{2n}\right] = 2\pi i.$$

For $2 \le k \le n-1$, we obtain

$$\rho_k = \frac{i}{n}\sum_{j=1}^{n}\sin\left(\frac{2\pi k(2j-1)}{2n}\right)\cot\left(\frac{(2j-1)\pi}{2n}\right)\left[2\sin^2\left(\frac{\pi(2j-1)}{2n}\right)+\cos\left(\frac{2\pi(2j-1)}{2n}\right)\right]$$
$$= \frac{i}{2n}\sum_{j=1}^{n}\cot\left(\frac{(2j-1)\pi}{2n}\right)\left[\sin\left(\frac{2\pi(k+1)(2j-1)}{2n}\right)+\sin\left(\frac{2\pi(k-1)(2j-1)}{2n}\right)\right]$$
$$= \frac{\rho_{k+1}}{2}+\frac{\rho_{k-1}}{2},$$

■

which implies that $\rho_{k+1} - \rho_k = \rho_k - \rho_{k-1} = \dots = \rho_2 - \rho_1 = 0$. Therefore, $\rho_k = 2\pi i$, $2 \le k \le n-1$. Using $\rho_k = -\rho_{2n-k}$, $1 \le k \le n$, we obtain the proof of Lemma 1.

Corollary 1. *The eigenvalues of C_2^h consist of 0 and $\pm 2\pi$.*

Corollary 2. *$(1/2)I + C_2^h$ is invertible, and $((1/2)I + C_2^h)^{-1}$ is uniformly bounded.*

Lemma 2. [25] *Let Y, Z be regular matrices of order m, and $X = Y + Z$. Then*

$$|\lambda(X) - \lambda_j(Z)| \le \max_{1\le j\le m}|\lambda_j(Y)|, 1 \le j \le m,$$

where $\lambda(X), \lambda(Z)$, and $\lambda(Y)$ are the eigenvalues of matrices X, Z, and Y, respectively. In particular, if a complex number β does not satisfy:

$$|\beta - \lambda_j(Z)| \le \max_{1\le j\le m}|\lambda_j(Y)|, 1 \le j \le m,$$

then β is not an eigenvalue of matrix X.

Corollary 3. *(1)* $(1/2)I + C^h$ *is invertible and* $((1/2)I + C^h)^{-1}$ *is uniformly bounded.*

(2) $\left\{\left(\frac{1}{2}I + C^h\right)^{-1} M^h\right\}$ *is a collectively compact operator sequence and convergent to* $\left(\frac{1}{2}I + C\right)^{-1} M$*, that is,*

$$\left(\frac{1}{2}I + C^h\right)^{-1} M^h \stackrel{c.c}{\to} \left(\frac{1}{2}I + C\right)^{-1} M.$$

where $\stackrel{c.c}{\to}$ *means the collectively compact convergence.*

Proof. (1) We firstly have

$$\frac{1}{2}I + C^h = \frac{1}{2}\left(I + 2c_2 D^h C_2^h\right) = \frac{1}{2}D^h\left((D^h)^{-1} + 2c_2 C_2^h\right).$$

Next, we discuss the eigenvalues of $(D^h)^{-1} + 2c_2 C_2^h$. As

$$\frac{1}{a_1(t,t)} = 1 > \frac{1-2\nu}{1-\nu} \geq 2c_2 \max_{1\leq j\leq 4n} \left|\lambda_j\left(C_2^h\right)\right|,$$

for any real number $\alpha \in (0, \nu/(1-\nu))$, we have

$$\left|\frac{1}{a_1(t,t)} - \alpha\right| \geq 1 - \alpha > \frac{1-2\nu}{1-\nu} \geq 2c_2 \max_{1\leq j\leq 4n} \left|\lambda_j\left(C_2^h\right)\right|.$$

From Lemma 2, we obtain $\rho((D^h)^{-1} + 2c_2 C_2^h) > \nu/(1-\nu)$. It means that $\|((D^h)^{-1} + 2c_2 C_2^h)^{-1}\| \leq (1-\nu)/\nu$. Also as D^h is invertible and uniformly bounded, $(1/2)I + C^h$ is invertible and uniformly bounded.

(2) As the kernel of integral operator M_{ij}, $i, j = 1, 2$, is a continuous function, we obtain [26–28] that $\{M_{ij}^h\}$ is a collectively compact operator sequence and convergent to M_{ij}, that is,

$$M_{ij}^h \stackrel{c.c}{\to} M_{ij}.$$

Thus, we have $M^h \stackrel{c.c}{\to} M$. From Corollary 2, we also obtain that $\{(\frac{1}{2}I + C^h)^{-1}M^h\}$ is a collectively compact operator sequence and convergent to $(\frac{1}{2}I + C)^{-1}M$. The proof of Corollary 3 is completed. ∎

From Eq. (1) and Corollary 3, we find that $u^{(l)}$ is a trivial solution as $\lambda^{(l)} = 0$, and if $\lambda^{(l)} \neq 0$, we have $\lambda_h^{(l)} \neq 0$. Let $\gamma_h^{(l)} = 1/\lambda_h^{(l)}$ and also suppose that the eigenvalues of $(\frac{1}{2}I + C)^{-1}M$ and $(\frac{1}{2}I + C^h)^{-1}M^h$ do not include -1, then Eqs. (6) and (16) can be rewritten as follows: find $\gamma^{(l)}$ and $u^{(l)} \in V^{(0)}$ satisfying

$$\gamma^{(l)}u^{(l)} = Lu^{(l)}, \text{ with } \|u^{(l)}\|_{0,\Gamma}^2 = \int_0^{2\pi} |u^{(l)}(s)|^2|x'(s)|ds = 1, \tag{20}$$

and find $\gamma_h^{(l)}$ and $u_h^{(l)}$ satisfying

$$\gamma_h^{(l)}u_h^{(l)} = L^h u_h^{(l)}, \text{ with } h\sum_{i=1}^{2}\sum_{j=0}^{2n-1}\left(u_{ih}^{(l)}(t_j)\right)^2|x'(t_j)| = 1, \tag{21}$$

where $L^h = [I + (\frac{1}{2}I + C^h)^{-1}M^h]^{-1}(\frac{1}{2}I + C^h)^{-1}(A^h + B^h)$, and $L = [I + (\frac{1}{2}I + C)^{-1}M]^{-1}(\frac{1}{2}I + C)^{-1}(A + B)$, and the space $V^{(m)} = C^{(m)}[0, 2\pi] \times C^{(m)}[0, 2\pi]$, $m = 0, 1, 2, \ldots$.

Theorem 1. *The approximate operator sequence $\{L^h\}$ is an asymptotically compact sequence and convergent to L in $V^{(0)}$, that is,*

$$L^h \stackrel{a.c}{\to} L, \tag{22}$$

where $\stackrel{a.c}{\to}$ means asymptotically compact convergence.

Proof. As the kernels of B_0 and $B_{ij}(i, j = 1, 2)$ are continuous functions, we have [16, 21]

$$B_0^h \stackrel{c.c}{\to} B_0 \text{ and } B_{ij}^h \stackrel{c.c}{\to} B_{ij} \text{ in } C[0, 2\pi], \text{ as } n \to \infty.$$

Also as $a_n(t, \tau)$ is a continuous approximate of $a(t, \tau)$, the approximate operator $\{A_0^h\}$ is an asymptotically compact sequence and convergent to A_0, that is, $A_0^h \stackrel{a.c}{\to} A_0$ in $C[0, 2\pi]$, as $n \to \infty$. Then we have $A^h \stackrel{a.c}{\to} A$ and $B^h \stackrel{c.c}{\to} B$ in $V^{(0)}$. It implies that for any bounded sequence $\{y_m \in V^{(0)}\}$ there exists a convergent subsequence in $\{(A^h + B^h)y_m\}$. Without loss of generality, we assume $(A^h + B^h)y_m \to z$, as $m \to \infty$. From the properties of asymptotically compact convergence and quadrature rules [22, 24], we have

$$\begin{aligned}
&\left\| L^h y_m - \left[I + \left(\frac{1}{2}I + C\right)^{-1} M \right]^{-1} \left(\frac{1}{2}I + C\right)^{-1} z \right\| \\
&\leq \left\| \left[I + \left(\frac{1}{2}I + C^h\right)^{-1} M^h \right]^{-1} \left(\frac{1}{2}I + C^h\right)^{-1} \right\| \cdot \| (A^h + B^h)y_m - z \| \\
&+ \left\| \left[I + \left(\frac{1}{2}I + C^h\right)^{-1} M^h \right]^{-1} \left(\frac{1}{2}I + C^h\right)^{-1} [C^h - C + M^h - M] \right. \\
&\left. \cdot \left[I + \left(\frac{1}{2}I + C\right)^{-1} M \right]^{-1} \left(\frac{1}{2}I + C\right)^{-1} z \right\| \to 0, \text{ as } m \to \infty \text{ and } h \to 0,
\end{aligned} \tag{23}$$

where $\| \cdot \|$ is a norm. It shows that $\{L^h : V^{(0)} \to V^{(0)}\}$ is an asymptotically compact operator sequence. Moreover, we will show that $L^h \stackrel{a,c}{\to} L$, as $n \to \infty$. In fact, as $A^h + B^h \stackrel{a,c}{\to} A + B$ for $y \in V^{(0)}$, we obtain

$$\|(A^h + B^h)y - (A + B)y\| \to 0, \text{ as } h \to 0. \tag{24}$$

From Corollary 3 and quadrature rules [24], we derive

$$\|L^h y - Ly\| \leq \left\| \left[I + \left(\frac{1}{2}I + C^h\right)^{-1} M^h \right]^{-1} \left(\frac{1}{2}I + C^h\right)^{-1} \right\| \cdot \|(A^h + B^h)y$$

$$-(A+B)y\| + \left\| \left[I + \left(\frac{1}{2}I + C^h\right)^{-1} M^h \right]^{-1} \left(\frac{1}{2}I + C^h\right)^{-1} [C^h - C + M^h - M] \right.$$

$$\left. \left[I + \left(\frac{1}{2}I + C\right)^{-1} M \right]^{-1} \left(\frac{1}{2}I + C\right)^{-1} (A+B)y \right\| \to 0, \text{ as } h \to 0.$$

The proof of Theorem 1 is completed. ∎

Corollary 4. [6,23,24]. *Under the assumption of Theorem 1, we have*

$$\|(L^h - L)L\| \to 0 \text{ and } \|(L^h - L)L^h\| \to 0, \text{ as } h \to 0. \tag{25}$$

Theorem 2. *Suppose $u(s) \in V^{(2l)}$ and kernel $m_{ij}(t,s) \in C^{2l+1}[0, 2\pi]$, $i, j = 1, 2$, then we have the following asymptotic expansion*

$$(L^h - L)u(s) = \sum_{j=1}^{l-1} \psi_j(s) diag(h^{2j+1}, h^{2j+1}) + O(h^{2l}), \tag{26}$$

where $\psi_j(s) \in V^{(2l-j)}$, $j = 1, \cdots, l-1$, are functions independent of h.

Proof. From the asymptotic expansion of the error for quadrature rules [26], we derive

$$(A+B)u(t) - (A^h + B^h)u(t) = \sum_{j=1}^{l-1} \varphi_j(t)\text{diag}(h^{2j+1}, h^{2j+1}) + O(h^{2l}),$$

where $\varphi_j(t) = \frac{\varsigma'(-2j)}{(2j)!} u^{(2j)}(t) \in V^{(2l-j)}$, $j = 1, \cdots, l-1$, are functions independent of h.

We also have the remainder estimate of the midpoint rule for periodic functions

$$\max_{0 \le s \le 2\pi} |(M - M^h)\phi(s)| = \|(M - M^h)\phi\| = O(h^{2l}), \forall \phi \in V^{(2l)},$$

and the identity

$$L^h u - Lu = \left[I + \left(\frac{1}{2}I + C^h\right)^{-1} M^h \right]^{-1} \left(\frac{1}{2}I + C^h\right)^{-1} [(A^h + B^h)u$$

$$-(A+B)u] + \left[I + \left(\frac{1}{2}I + C^h\right)^{-1} M^h \right]^{-1} \left(\frac{1}{2}I + C^h\right)^{-1} [C^h - C$$

$$+M^h - M] \left[I + \left(\frac{1}{2}I + C\right)^{-1} M \right]^{-1} \left(\frac{1}{2}I + C\right)^{-1} (A+B)u.$$

According to the above equations and Eq. (15), when we let $\psi_j(s) = [I + (\frac{1}{2}I + C^h)^{-1} M^h]^{-1} (\frac{1}{2}I + C^h)^{-1} \varphi_j(s)$, we complete the proof of Theorem 2. ∎

We know that if $\gamma^{(l)}$ is an isolated eigenvalue of Eq. (20), then the dimension of its eigenspace is finite [29] and the complex conjugate $\bar{\gamma}^{(l)}$ of $\gamma^{(l)}$ is also an eigenvalue of the conjugate operator $\bar{L}$. Let $\bar{V}_\gamma = \text{span}\{\bar{u}^{(l)}_{(1)}, \cdots, \bar{u}^{(l)}_{(\chi)}\}$ and $V_\gamma = \text{span}\{u^{(l)}_{(1)}, \cdots, u^{(l)}_{(\chi)}\}$ be the eigenspace of $\bar{L}$ and L, respectively, which constructs the biorthogonal system

$$\langle u^{(l)}_{(i)}, \bar{u}^{(l)}_{(j)} \rangle = \delta_{ij}, \quad i, j = 1, \cdots, \chi, \tag{27}$$

with $\|u^{(l)}_{(i)}\| = 1, i = 1, \cdots, \chi$. Let $\gamma^{(l)}_h$ and $V_{\gamma h}$ be the eigenvalue and the eigenspace of L^h, corresponding to $\gamma^{(l)}$ and V_γ, respectively. There exists $\dim V_{\gamma h} = \chi_1 \le \dim V_\gamma \le \chi$ (see [29]). Assume that $\{u^{(l)}_{(i)h}\}$ and $\{\bar{u}^{(l)}_{(i)h}\}$ are the approximate eigenvectors of $\{u^{(l)}_{(i)}\}$ and $\{\bar{u}^{(l)}_{(i)}\}, i = 1, \cdots, \chi_1$, which satisfy the following normalized conditions

$$\begin{cases} \langle u^{(l)}_{(i)h}, \bar{u}^{(l)}_{(j)h} \rangle = \delta_{ij}, & i, j = 1, \cdots, \chi_1, \\ \langle u^{(l)}_{(i)h}, \bar{u}^{(l)}_{(i)} \rangle = 1, & i = 1, \cdots, \chi_1. \end{cases} \tag{28}$$

Theorem 3. *Under the hypotheses of Corollary 4, Eqs. (27) and (28), we obtain*

$$\begin{cases} |\gamma^{(l)}_h - \gamma^{(l)}| = O(\|L(L - L^h)\|), \\ \|u^{(l)}_{(i)} - u^{(l)}_{(i)h}\| = O(\|L(L - L^h)\|). \end{cases} \tag{29}$$

Proof. As we have the following equation

$$\begin{aligned} (\gamma^{(l)} I - L)\big(u^{(l)}_{(i)} - u^{(l)}_{(i)h}\big) &= -\gamma^{(l)} u^{(l)}_{(i)h} + L u^{(l)}_{(i)h} \\ &= \big(\gamma^{(l)}_h - \gamma^{(l)}\big) u^{(l)}_{(i)h} + \big(L u^{(l)}_{(i)h} - L^h u^{(l)}_{(i)h}\big), \end{aligned} \tag{30}$$

taking the inner product by $\bar{u}^{(l)}_i$ on the both sides of the identity and considering the inner product property of Eqs. (27) and (28), we obtain

$$\begin{aligned} 0 &= \big\langle (\gamma^{(l)} I - L)\big(u^{(l)}_{(i)} - u^{(l)}_{(i)h}\big), \bar{u}^{(l)}_i \big\rangle \\ &= \big(\gamma^{(l)}_h - \gamma^{(l)}\big)\big\langle u^{(l)}_{(i)h}, \bar{u}^{(l)}_{(i)} \big\rangle + \big\langle L u^{(l)}_{(i)h} - L^h u^{(l)}_{(i)h}, \bar{u}^{(l)}_{(i)} \big\rangle \\ &= \big(\gamma^{(l)}_h - \gamma^{(l)}\big) + \big\langle L(L - L^h) u^{(l)}_{(i)h}, \bar{u}^{(l)}_{(i)} \big\rangle / \bar{\gamma}^{(l)}, \end{aligned}$$

that is

$$|\gamma^{(l)}_h - \gamma^{(l)}| = O(\|L(L - L^h)\|). \tag{31}$$

Define the subspace

$$V^\perp_\gamma = \{v : \langle v, \bar{u}^{(l)}_{(i)} \rangle = 0, i = 1, \cdots, \chi\}.$$

Obviously, under the restriction to the subspace $V^\perp_{\gamma^{(l)}}$, $(\gamma^{(l)} I - L)^{-1}$ exists. As $(u^{(l)}_{(i)} - u^{(l)}_{(i)h}) \in V^\perp_{\gamma^{(l)}}$, from Eq. (30), we deduce that there exists a constant $c > 0$ satisfying

$$\begin{aligned} c\|u^{(l)}_{(i)} - u^{(l)}_{(i)h}\| &\le \|(\gamma^{(l)} I - L)\big(u^{(l)}_{(i)} - u^{(l)}_{(i)h}\big)\| \\ &\le |\gamma^{(l)}_h - \gamma^{(l)}|\|u^{(l)}_{(i)h}\| + \|(L - L^h)L^h\|\|u^{(l)}_{(i)h}\|/|\bar{\gamma}^{(l)}|. \end{aligned} \tag{32}$$

As we have $\|u^{(l)}_{(i)h}\| \leq \|u^{(l)}_{(i)} - u^{(l)}_{(i)h}\| + \|u^{(l)}_{(i)}\|$ with Eqs. (31) and (32), we complete the proof of this theorem. ∎

Corollary 5. *Suppose* $\{\gamma^{(l)}_h, u^{(l)}_{(i)h}\}$ *are the eigenvalue and eigenvector of Eq. (21) and* $u^{(l)}_{(i)}$, $\bar{u}^{(l)}_{(i)} \in V^{(2)}$, $i = 1, \cdots, \chi_1$, *then*

$$\left|\gamma^{(l)}_h - \gamma^{(l)}\right| = O(h^2), \quad \left\|u^{(l)}_{(i)} - u^{(l)}_{(i)h}\right\| = O(h^2), \tag{33}$$

and

$$\left\|\bar{u}^{(l)}_{(i)} - \bar{u}^{(l)}_{(i)h}\right\| = O(h^2). \tag{34}$$

Proof. The proof of Eq. (33) can be found in [23,29]. As Eq. (34), we define the subspace

$$\bar{V}^{\perp}_{\gamma^{(l)}_h} = \left\{v \in L^2[0, 2\pi] : \langle v, u^{(l)}_{(i)h}\rangle = 0, i = 1, \cdots, \chi_1\right\}.$$

Note that under the restriction to the subspace $\bar{V}^{\perp}_{\gamma^{(l)}_h}$, $(\bar{\gamma}^{(l)} I - \bar{L}^h)^{-1}$ exists and is uniformly bounded. Following Eq. (28) and $\bar{u}^{(l)}_{(i)} - \bar{u}^{(l)}_{(i)h} \in \bar{V}^{\perp}_{\gamma^{(l)}_h}$, then

$$\begin{aligned}
&\left(\bar{\gamma}^{(l)}_h I - \bar{L}^h\right)\left(\bar{u}^{(l)}_{(i)} - \bar{u}^{(l)}_{(i)h}\right) = \bar{\gamma}^{(l)}_h \bar{u}^{(l)}_{(i)} - \bar{L}^h \bar{u}^{(l)}_{(i)} \\
&= \frac{\bar{\gamma}^{(l)}_h}{\bar{\gamma}^{(l)}} \bar{L}\bar{u}^{(l)}_{(i)} - \bar{L}^h \bar{u}^{(l)}_{(i)} = \frac{1}{\bar{\gamma}^{(l)}} \left(\frac{\bar{\gamma}^{(l)}_h}{\bar{\gamma}^{(l)}} \bar{L} - \bar{L}^h\right) \bar{L}\bar{u}^{(l)}_{(i)} \\
&= \frac{1}{\bar{\gamma}^{(l)}} (\bar{L} - \bar{L}^h)\bar{L}\bar{u}^{(l)}_{(i)} + O(h^2) = O(h^2),
\end{aligned}$$

thus, (34) is true. ∎

III. ASYMPTOTIC ERROR EXPANSIONS AND RICHARDSON EXTRAPOLATION

Theorem 4. *Under the hypotheses of Corollary 5 and Theorem 3, if* $\{\gamma^{(l)}, u^{(l)}_{(i)}\}$ *and* $\{\gamma^{(l)}_h, u^{(l)}_{(i)h}\}$ *are the eigenvalue and eigenvector of Eqs. (20) and (21), respectively, then there exist a constant* d_1 *and vector functions* $w_i \in V^{(3)}, i = 1, \cdots, \chi_1$, *independent of h, such that*

$$\gamma^{(l)}_h - \gamma^{(l)} = d_1 h^3 + O(h^5), \tag{35}$$

$$u^{(l)}_{(i)h} - u^{(l)}_{(i)} = w_i h^3 + O(h^5). \tag{36}$$

Proof. From Theorem 2, we obtain

$$\begin{aligned}
&L^h\left(u^{(l)}_{(i)} + w_i h^3\right) - (\gamma^{(l)} + d_1 h^3)\left(u^{(l)}_{(i)} + w_i h^3\right) \\
&= (L^h - L)u^{(l)}_{(i)} + h^3\left(L^h w_i - d_1 u^{(l)}_{(i)} - \gamma^{(l)} w_i\right) - d_1 w_i h^6 \\
&= h^3\left(L^h w_i - d_1 u^{(l)}_{(i)} - \gamma^{(l)} w_i + \psi\right) + O(h^5).
\end{aligned} \tag{37}$$

Choose the constant d_1 and function w_i satisfy the following operator equations:

$$L^h w_i - \gamma^{(l)} w_i = d_1 u_{(i)}^{(l)} - \psi,$$
$$\langle d_1 u_{(i)}^{(l)} - \psi, \phi \rangle = 0, \forall \phi \in \bar{V}_{\gamma^{(l)}}^{\perp}. \tag{38}$$

Obviously, under the restriction condition of Eq. (38), there exists a unique solution w_i in Eq. (38). Taking $\phi = \bar{u}_{(i)}^{(l)}$, we obtain

$$d_1 = \langle \psi, \bar{u}_{(i)}^{(l)} \rangle.$$

Thus, Eq. (37) is converted to be

$$L^h\big(u_{(i)}^{(l)} + w_i h^3\big) - (\gamma^{(l)} + d_1 h^3)\big(u_{(i)}^{(l)} + w_i h^3\big) = O(h^5). \tag{39}$$

As $\{\gamma_h^{(l)}, u_{(i)h}^{(l)}\}$ satisfies

$$L^h u_{(i)h}^{(l)} - \gamma_h^{(l)} u_{(i)h}^{(l)} = 0, \tag{40}$$

and from Eqs. (39) and (40), we obtain

$$L^h\big(u_{(i)h}^{(l)} - u_{(i)}^{(l)} - w_i h^3\big) - \gamma_h^{(l)}\big(u_{(i)h}^{(l)} - u_{(i)}^{(l)} - w_i h^3\big)$$
$$- \big(\gamma_h^{(l)} - \gamma^{(l)} - d_1 h^3\big)\big(u_{(i)}^{(l)} + w_i h^3\big) = O(h^5). \tag{41}$$

We also have

$$\langle u_{(i)}^{(l)}, \bar{u}_{(i)h}^{(l)} \rangle = \langle u_{(i)}^{(l)}, \bar{u}_{(i)}^{(l)} \rangle + \langle u_{(i)}^{(l)}, \bar{u}_{(i)h}^{(l)} - \bar{u}_{(i)}^{(l)} \rangle$$
$$= 1 + \langle u_{(i)}^{(l)} - u_{(i)h}^{(l)}, \bar{u}_{(i)h}^{(l)} - \bar{u}_{(i)}^{(l)} \rangle = 1 + O(h^5). \tag{42}$$

Taking the inner product on both sides of Eq. (41) by $\bar{u}_{(i)h}^{(l)}$ and using Eqs. (28) and (42), we obtain

$$\gamma_h^{(l)} - \gamma^{(l)} - d_1 h^3 = O(h^5). \tag{43}$$

Substituting Eq. (43) into Eq. (41), we have

$$\big(L^h - \gamma_h^{(l)} I\big)\big(u_{(i)h}^{(l)} - u_{(i)}^{(l)} - w_i h^3\big) = O(h^5). \tag{44}$$

Obviously, under the restriction in the invariant subspace

$$V_{\gamma_h^{(l)}}^{\perp} = \big\{v : \langle v, \bar{u}_{(i)h}^{(l)} \rangle = 0, i = 1, \cdots, \chi_1\big\},$$

the operator $(L^h - \gamma_h^{(l)} I)$ is invertible and $(L^h - \gamma_h^{(l)} I)^{-1}$ is uniformly bounded. Generally, $u_{(i)h}^{(l)} - u_{(i)}^{(l)} - w_i h^3 = g \notin V_{\gamma_h^{(l)}}^{\perp}$, but $g - P^h g \in V_{\gamma_h^{(l)}}^{\perp}$, where

$$P^h g = \sum_{i=1}^{\chi_1} \langle g, \bar{u}_{(i)h}^{(l)} \rangle u_{(i)h}^{(l)} \tag{45}$$

is a projection of g on $V_{\gamma(l)}$. From Eq. (42), we obtain the estimate

$$\left|\langle g, \bar{u}^{(l)}_{(i)h}\rangle\right| = \left|\langle g, \bar{u}^{(l)}_{(i)h} - \bar{u}^{(l)}_{(i)}\rangle\right| \le \left|\langle u^{(l)}_{(i)h} - u^{(l)}_{(i)}, \bar{u}^{(l)}_{(i)h} - \bar{u}^{(l)}_{(i)}\rangle\right| + \left|\langle w_i, \bar{u}^{(l)}_{(i)h} - \bar{u}^{(l)}_{(i)}\rangle\right| h^3 = O(h^5),$$

which means $\|P^h g\| = O(h^5)$. However, from Eq. (44), we have

$$O(h^5) = \left\|\left(L^h - \gamma^{(l)}_h I\right)g\right\| \ge \left\|\left(L^h - \gamma^{(l)}_h I\right)(g - P^h g)\right\| - O(h^5) \ge a\|g - P^h g\| - O(h^5),$$

where a is a constant, that is, $\|g - P^h g\| = O(h^5)$. Therefore, we obtain $g = u^{(l)}_{(i)h} - u^{(l)}_{(i)} - w_i h^3 = O(h^5)$, which completes the proof of Theorem 4. ∎

Corollary 6. *Under the hypotheses of Theorem 4, there exist constants d_1, d_2 and vector functions $w_{i1}, w_{i2} \in V^{(5)}, i = 1, \cdots, \chi_1$, independent of h, such that*

$$\lambda^{(l)}_h - \lambda^{(l)} = d_1 h^3 + d_2 h^5 + O(h^7), \tag{46}$$

$$u^{(l)}_{(i)h} - u^{(l)}_{(i)} = w_{i1} h^3 + w_{i2} h^5 + O(h^7). \tag{47}$$

By means of the Richardson EAs for the asymptotic expansions, we can obtain the approximate solutions with a higher order accuracy $O(h^5)$ by solving some coarse grid discrete equations in parallel. The EAs are described as follows.

Step 1. Choose the mesh widths h and $h/2$ to calculate the solutions of Eq. (16) as $(\lambda^{(i)}_h, u^{(i)}_h)$ and $(\lambda^{(i)}_{h/2}, u^{(i)}_{h/2})$.

Step 2. Calculating the h^3 -Richardson extrapolations based on the asymptotic expansions Eqs. (35) and (36), we obtain

$$\begin{cases} \lambda^{(i)*}_h = \left(8\lambda^{(i)}_{h/2} - \lambda^{(i)}_h\right)/7, \\ u^{(i)*}_h(s_j) = \left(8u^{(i)}_{h/2}(s_j) - u^{(i)}_h(s_j)\right)/7, \end{cases} \tag{48}$$

where the error estimates are $|\lambda^{(i)*}_h - \lambda^{(i)}| = O(h^5)$ and $\|u^{(i)*}_h(s_j) - u^{(i)}(s_j)\| = O(h^5)$ with $s_j = jh,\ j = 1, \cdots, 2n$.

Step 3. From the asymptotic expansions Eqs. (35) and (36), we also obtain

$$\left|\lambda^{(i)}_{h/2} - \lambda^{(i)}\right| \le \left|8/7\lambda^{(i)}_{h/2} - 1/7\lambda^{(i)}_h - \lambda^{(i)}\right| +1/7\left|\lambda^{(i)}_{h/2} - \lambda^{(i)}_h\right| \le 1/7\left|\lambda^{(i)}_{h/2} - \lambda^{(i)}_h\right| + O(h^5), \tag{49}$$

and

$$\left\|u^{(i)}_{h/2}(s_j) - u^{(i)}(s_j)\right\| \le \left\|8/7u^{(i)}_{h/2}(s_j) - 1/7u^{(i)}_h(s_j) - u^{(i)}(s_j)\right\| +1/7\left\|u^{(i)}_{h/2}(s_j) - u^{(i)}_h(s_j)\right\| \le 1/7\left\|u^{(i)}_{h/2}(s_j) - u^{(i)}_h(s_j)\right\| + O(h^5). \tag{50}$$

Note that the inequalities can be used to construct self-adaptive algorithms.

IV. FOURIER EXPANSION IN EIGENSOLUTIONS

Some applications have been shown in some papers [14–17] for Steklov eigensolutions. Next, we will obtain the Fourier expansion in eigensolutions and use the expansion to provide the application for solving elasticity problems.

Theorem 5. [1,2] *(1) The eigenvalues are real,all non-zero eigenvalues are positive, and the eigenvalues form an increasing sequence to infinity:* $\lambda_1 \leq \ldots \leq \lambda_n \leq \ldots$
(2) The sequence of eigenvectors are boundary orthonormal and complete, that is

$$\int_\Gamma u^{(m)} \cdot u^{(l)} ds = \delta_{ml}, m, l = 1, 2, \ldots.$$

where δ_{ml} *is the Kronecker delta.*

We obtain the result that there is a generalized Fourier orthonormal basis and L_2-functions can be expanded as discrete Fourier series. The displacement u on Γ can be expanded

$$u = \sum_{l=1}^{\infty} q_l u^{(l)}, \text{ on } \Gamma. \tag{51}$$

From the gradient of deformation, we obtain

$$\frac{\partial u}{\partial n} = \sum_{l=1}^{\infty} q_l \lambda^{(l)} u^{(l)}, \text{ on } \Gamma. \tag{52}$$

Dirichlet problem: Assume the displacement $u = f(x)$ on Γ is prescribed, where $f(x)$ is a L_2-function. Multiplying the both sides of Eq. (51) by $u^{(l)}$ to integral, and using orthonormality, we obtain the coefficient q_l as

$$q_l = \int_\Gamma f \cdot u^{(l)} ds. \tag{53}$$

Substituting q_l into Eq. (52), we obtain the traction $t = \partial u / \partial n$ on Γ.

Neumann problem: Assume the traction $t = g(x)$ on Γ is prescribed, where $g(x)$ is a L_2-function satisfying the compatibility conditions $\int_\Gamma g(x) ds = 0$. Using orthonormality and Eq. (42), we obtain the coefficient q_l as

$$q_l = \frac{1}{\lambda^{(l)}} \int_\Gamma g \cdot u^{(l)} ds \qquad \lambda^{(l)} \neq 0.$$

Substituting q_l into Eq. (51), we obtain the displacement u on Γ.

Once eigensolutions are known, we can obtain the solutions u and $\partial u / \partial n$ on Γ of elasticity problems. So the displacement vector and stress tensor in Ω can be calculated [5–7] by

$$\begin{cases} u_i(y) = \int_\Gamma h_{ij}^*(y, x) t_j(x) ds_x - \int_\Gamma k_{ij}^*(y, x) u_j(x) ds_x, & \forall y \in \Omega, \\ \sigma_{ij}(y) = \int_\Gamma h_{ijl}^*(y, x) t_l(x) ds_x - \int_\Gamma k_{ijl}^*(y, x) u_l(x) ds_x, & \forall y \in \Omega, \end{cases} \tag{54}$$

where

$$\begin{cases} k^*_{ijl} = [(1-2\nu)(r_{,j}\delta_{li} + r_{,i}\delta_{lj} - r_{,l}\delta_{ij}) + 2r_{,i}r_{,j}r_{,l}]/[4\pi(1-\nu)r], \\ h^*_{ijl} = \frac{\mu}{2\pi(1-\nu)r^2}\left\{2\frac{\partial r}{\partial n}[(1-2\nu)r_{,l}\delta_{ij} + \nu(r_{,j}\delta_{il} + r_{,i}\delta_{jl}) - 4r_{,i}r_{,j}r_{,l}] + 2\nu(n_i \right. \\ \left. \cdot r_{,j}r_{,l} + n_j r_{,i}r_{,l}) + (1-2\nu)(2n_l r_{,j}r_{,i} + n_j\delta_{il} + n_i\delta_{jl}) - (1-4\nu)n_l\delta_{ij}\right\}. \end{cases}$$

From the generalized Fourier expansion, we obtain that $\|u_h - u\| = O(h^3)$ and $\|t_h - t\| = O(h^3)$ on Γ. The h^3-Richardson extrapolation also can be used for the displacement u_h and the traction t_h on Γ,

$$\tilde{u}_h(y) = (8u_{h/2}(y) - u_h(y))/7, \quad \tilde{t}_h(y) = (8t_{h/2}(y) - t_h(y))/7, \quad y \in \Gamma, \tag{55}$$

which have the error estimates: $\|\tilde{u}_h - u\| = O(h^5)$ and $\|\tilde{t}_h - t\| = O(h^5)$, respectively.

So the computational process of the displacement vector $u = (u_1, u_2)$ and stress tensor σ_{ij} in Ω can be prescribed as follows:

a. Solve $u^{(l)}$ and $\lambda^{(l)}$ from the singular integral Eq. (16);
b. Compute u and t on Γ by the generalized Fourier series (51) and (52);
c. Calculate the displacement vector u and stress tensor σ_{ij} in Ω following Eq. (54).

Remark 1. The Fourier expansions present an entirely new way to solve the elastic boundary value problems by the discrete Fourier analysis, which also provides a tool to research the solutions of Stokes equations and problems in vibration theory with Dirichlet, Neumann, and mixed boundary conditions.

V. NUMERICAL EXAMPLE

Example 1. Consider a circular isotropic elastic body with radius a under plane strain deformation. Parton and Perlin [16] presented some analytic solutions about eigenvalues for this problem as follows:

$$\lambda_l = \frac{2\mu l}{a}, \; l = 1, 2, \cdots. \tag{56}$$

Consider the circular plane strain deformation problem with radius $a = 1$ and material properties $\mu = 0.25$ and $\nu = 2.5$. Let $h = \pi/n (n \in N)$ be the mesh width and $s_j = jh (j = 0, 1, \cdots, 2n-1)$ be the nodes. We have computed numerical eigensolutions in article [20]. As the first non-zero eigenvalue plays an important role [30–34], we show the errors in Fig. 1. For the x-axis, we set $x = \log_2 n$. We can find that the numerical eigenvalue rapidly approximate the first non-zero analytic eigenvalue $\lambda_1 = 0.5$.

Neumann boundary condition 1: As the eigensolutions have been obtained, we use the Fourier expansion to solve the Neumann problem and to compute the displacement vector u on Γ.

Set a Neumann boundary condition: $t = (t_1, t_2)$ with $t_1 = 2\cos(3\theta) + 2\cos(2\theta)\cos\theta$ and $t_2 = 2\sin(3\theta) + 2\sin(2\theta)\cos\theta (\theta \in [0, 2\pi])$. The displacement u on Γ can now be calculated. In Table I, we list the errors of the displacement $u_h(y) = (u_{1h}, u_{2h})$ on Γ computed following formulas (51) and (52), where $e^h_{vi}(\theta) = |u_{ih}(\theta) - u_i(\theta)|$, $i = 1, 2$, and $\tilde{e}^h_{vi}(\theta) = |\tilde{u}_{ih}(\theta) - u_i(\theta)|$ is the error after using the EAs, and $p^h_{vi}(\theta)$ is the a posterior error estimate.

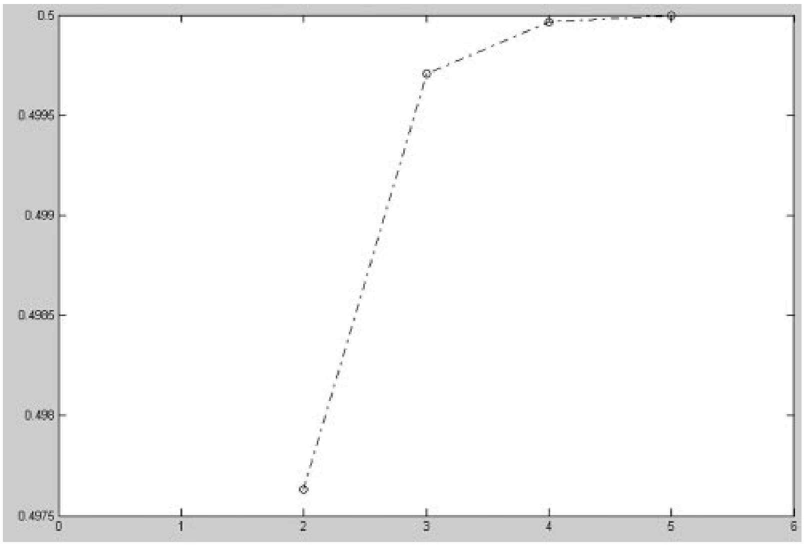

FIG. 1. The first non-zero numerical eigenvalue $\lambda_1 = 0.5$.

From Table I, we can numerically estimate $e^h_{vi}(\theta)/e^{h/2}_{vi}(\theta) \approx 2^3, \tilde{e}^h_{vi}(\theta)/\tilde{e}^{h/2}_{vi}(\theta) \approx 2^5$. So we obtain the result that the approximate solutions are $O(h^3)$ convergent rate. After Richardson extrapolation, the approximate solutions are $O(h^5)$ convergent rate.

Below, we calculate the displacement $u_h(y) = (u_{1h}, u_{2h})$ in Ω following formulae Eq. (54), where the inner point is $P = (0.25, 0.5)$. Let $e^h_{vi}(P) = |u_i(P) - u_{ih}(P)|$, $i = 1, 2$, and $\tilde{e}^h_{vi}(P)$ be the error after the EAs, and $p^h_{vi}(P)$ be the corresponding a posteriori error estimate.

From Table II, we have $e^h_{vi}(P)/e^{h/2}_{vi}(P) \approx 2^3$, $\tilde{e}^h_{vi}(P)/\tilde{e}^{h/2}_{vi}(P) \approx 2^5$. So the convergence rates of the solutions in Ω are still $O(h^3)$ and $O(h^5)$, respectively.

Neumann linear boundary condition 2: A class of linear boundary condition $\frac{\partial u}{\partial n} = cu + g(x)$ is considered after finding the eigensolutions. We set that $c = 1$ and $g(x) = (g_1, g_2)$ with $g_1 = 2\cos(3\theta) + 2\cos(2\theta)\cos\theta$ and $g_2 = 2\sin(3\theta) + 2\sin(2\theta)\cos\theta(\theta \in [0, 2\pi])$ to calculate the displacement on Γ. In Table III we list the errors of the displacement $u_h(y) = (u_{1h}, u_{2h})$ on Γ computed by formulae (51) and (52), where notations are used as Table I.

TABLE I. The errors analysis of $(u_{1h}(\theta), u_{2h}(\theta))$ on Γ.

n	24	48	96	192	384
$e^h_{v1}(0)$	8.19E − 04	1.02E − 04	1.27E − 05	1.59E − 06	1.99E − 07
$\tilde{e}^h_{v1}(0)$		6.52E − 07	2.02E − 08	6.30E − 10	1.98E − 11
$p^h_{v1}(0)$		1.03E − 04	1.27E − 05	1.59E − 06	1.99E − 07
$e^h_{v1}(\pi/4)$	5.50E − 04	6.83E − 05	8.53E − 06	1.07E − 06	1.33E − 07
$\tilde{e}^h_{v1}(\pi/4)$		4.59E − 07	1.42E − 08	4.43E − 10	1.34E − 11
$p^h_{v1}(\pi/4)$		6.88E − 05	8.54E − 06	1.07E − 06	1.33E − 07
$e^h_{v2}(\pi/4)$	5.79E − 04	7.20E − 05	8.99E − 06	1.12E − 06	1.40E − 07
$\tilde{e}^h_{v2}(\pi/4)$		4.61E − 07	1.43E − 08	4.46E − 10	1.35E − 11
$p^h_{v2}(\pi/4)$		7.25E − 05	9.00E − 06	1.12E − 06	1.40E − 07
$e^h_{v2}(3\pi/4)$	7.77E − 04	9.66E − 05	1.21E − 05	1.51E − 06	1.88E − 07
$\tilde{e}^h_{v2}(3\pi/4)$		6.49E − 07	2.01E − 08	6.27E − 10	1.91E − 11
$p^h_{v2}(\pi/4)$		9.73E − 05	1.21E − 05	1.51E − 06	1.88E − 07

TABLE II. The errors analysis of (u_{1h}, u_{2h}) in Ω at $P = (0.25, 0.5)$.

n	24	48	96	192	384
$e^h_{v1}(P)$	1.07E − 04	1.33E − 05	1.66E − 06	2.07E − 07	2.59E − 08
$\tilde{e}^h_{v1}(P)$		8.72E − 08	2.70E − 09	8.42E − 11	2.55E − 12
$p^h_{v1}(P)$		1.34E − 05	1.66E − 06	2.07E − 07	2.59E − 08
$e^h_{v2}(P)$	2.66E − 04	3.30E − 05	4.12E − 06	5.15E − 07	6.44E − 08
$\tilde{e}^h_{v2}(P)$		2.17E − 07	6.71E − 09	2.09E − 10	6.33E − 12
$p^h_{v2}(P)$		3.32E − 05	4.13E − 06	5.15E − 07	6.44E − 08

TABLE III. The errors of $(u_{1h}(\theta), u_{2h}(\theta))$ on Γ when $\theta_1 = 0, \theta_2 = \pi/4$

n	24	48	96	192	384
$e^h_{v1}(0)$	1.19E − 03	1.46E − 04	1.81E − 05	2.25E − 06	2.80E − 07
$\tilde{e}^h_{v1}(0)$		3.52E − 06	1.09E − 07	3.39E − 9	1.06E − 10
$p^h_{v1}(0)$		1.53E − 04	1.91E − 05	2.27E − 06	2.81E − 07
$e^h_{v2}(\pi/4)$	3.50E − 03	4.31E − 04	5.30E − 05	6.58E − 06	8.21E − 07
$\tilde{e}^h_{v2}(\pi/4)$		7.53E − 06	2.32E − 07	7.21E − 9	2.25E − 10
$p^h_{v2}(\pi/4)$		4.42E − 04	5.34E − 05	6.59E − 06	8.21E − 07

From Table III, we can see the convergence numerically $\log_2(e^h_{vi}(\theta)/e^{h/2}_{vi}(\theta)) \approx 3$. It means that the convergence rates of the solutions are $O(h^3)$. Other boundary conditions for elasticity can be similarly obtained. So the numerical algorithms are very effective for calculating the elasticity.

VI. CONCLUSIONS

The theory of fundamental boundary eigensolutions gives a new point of view for studying the numerical solution of elastostatic boundary value problems. The following conclusions can be drawn:

Numerical results show that the methods not only have high accuracy, but also can be applied with the h^3-Richardson EA to reach higher accuracy. According to the numerical results, we find that the larger the scale of the problems is the more precision the EAs obtains.

The advantages for the methods are as follows: computing entry of discrete matrixes is very simple and straightforward, without any singular integrals. The methods are high accuracy algorithms of $O(h^3)$ and the EAs are $O(h^5)$ convergence rates, respectively.

In this article, the elastic problems are solved indirectly. Harnessing the generalized Fourier series, only the coefficients of the series need to be calculated.

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