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Mechanical quadrature methods and extrapolation for solving nonlinear boundary Helmholtz integral equations*

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Abstract This paper presents mechanical quadrature methods (MQMs) for solving nonlinear boundary Helmholtz integral equations. The methods have high accuracy of order $O(h^3)$ and low computation complexity. Moreover, the mechanical quadrature methods are simple without computing any singular integration. A nonlinear system is constructed by discretizing the nonlinear boundary integral equations. The stability and convergence of the system are proved based on an asymptotical compact theory and the Stepleman theorem. Using the h^3-Richardson extrapolation algorithms (EAs), the accuracy to the order of $O(h^5)$ is improved. To slove the nonlinear system, the Newton iteration is discussed extensively by using the Ostrowski fixed point theorem. The efficiency of the algorithms is illustrated by numerical examples.

Key words Helmholtz equation, mechanical quadrature method, Newton iteration, nonlinear boundary condition

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1 Introduction

Time-harmonic acoustic wave scattering or radiation by a cylindrical obstacle is essentially a two-dimensional problem, and is often described in acoustic media by the Helmholtz equation with associated boundary conditions. In this paper, we consider the following nonlinear boundary value problems:

$$\begin{cases} \Delta\tilde{u} - \alpha^2\tilde{u} = 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial\tilde{u}}{\partial n} = -G(\tilde{u}) + f(x) = -g(x,\tilde{u}) + f(x) \quad \text{on} \quad \Gamma, \end{cases} \tag{1}$$

where α is a real and positive constant and called the acoustic wave number, $\Omega \subset R^2$ is a bounded, simply connected domain with a smooth boundary Γ, $\frac{\partial}{\partial n}$ is an outward normal derivative on Γ, and $g(x,\tilde{u})$ and $f(x)$ are given functions.

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A considerable part of researches on the numerical solution of the Helmholtz equation were concerned with the applications of finite element methods[1–5], such as Hiptmair and Meury[2] introduced the coupled boundary element finite element schemes for its numerical simulation, and Sze and Liu[5] gave the hybrid-Trefftz triangular finite element methods. Ruotsalainen and Wendland[6] constructed boundary integral equation for the nonlinear boundary condition and used Galerkin approach to obtain the solution with the accuracy order $O(h^2)$. Atkinson and Chandler[7] used quadrature method to obtain high accuracy solutions for Laplace's nonlinear boundary integral equations. Huang and Wang[8] constructed mechanical quadrature methods (MQMs) for Helmholtz equation with mixed linear boundary condition, which have $O(h^3)$ order of convergent rates.

Using Green's representation formula for Helmholtz's equation[8–9], the function satisfies

$$\tilde{u}(y)=\int_\Gamma k^*(y,x)\tilde{u}(x)\mathrm{d}s_x-\int_\Gamma h^*(y,x)\frac{\partial\tilde{u}}{\partial n}\mathrm{d}s_x,\quad y\in\Omega. \tag{2}$$

Letting y tend to a point on Γ by the potential theory, and using the boundary condition of Eq. (1), we obtain

$$\begin{aligned}&\frac{1}{2}\tilde{u}(y)-\int_\Gamma k^*(y,x)\tilde{u}(x)\mathrm{d}s_x-\int_\Gamma h^*(y,x)G(\tilde{u})\mathrm{d}s_x\\&=-\int_\Gamma h^*(y,x)f(x)\mathrm{d}s_x,\quad y\in\Gamma,\end{aligned} \tag{3}$$

where $x=(x_1,x_2), y=(y_1,y_2)$, $|x-y|^2=(x_1-y_1)^2+(x_2-y_2)^2$, and $h^*(y,x)$ is the fundamental solution of the Helmholtz equation[10],

$$h^*(y,x)=\begin{cases}-\dfrac{1}{2\pi}\mathrm{K}_0(\alpha|x-y|)\quad\text{for}\quad\alpha>0,\\-\dfrac{1}{2\pi}\log|x-y|\quad\text{for}\quad\alpha=0,\end{cases} \tag{4}$$

where K_0 is the modified Bessel function

$$\mathrm{K}_0(z)=-\log z+\log 2-\gamma,\quad z\to 0 \tag{5}$$

with Euler's constant $\gamma=0.577\,21\cdots$, and $k^*(y,x)=\frac{\partial h^*(y,x)}{\partial n}$. The boundary value $\tilde{u}$ on Γ can be solved by Eq. (3), and then the normal derivative can be obtained from Eq. (1). So $\tilde{u}(y)$ will be calculated following Eq. (2) for all $y\in\Omega$.

Some numerical methods are proposed for the nonlinear integral equations (3). Lin[11] researched nonlinear integral equations with smooth kernels. Ismail[12] described a special quadrature method for two-dimensional nonlinear singular integral equation with Hilbert kernel, which is not convenient to construct extrapolation algorithms. Yan[13] combined a quadrature method for boundary integral equations with a preconditioned iterative method for solving a two-dimensional Helmholtz equation. Shen and Wang[14] performed a complete error analysis for spectral-Galerkin approximation of a model Helmholtz equation with high wave numbers. Kress and Sloan[15] gave a quadrature method for solving the boundary integral equations, and their error analysis was based on trigonometric interpolation theory. Cheng et al.[16] used mechanical quadrature methods for solving a Laplace equation with nonlinear boundary condition.

We present the mechanical quadrature methods (MQMs) of boundary integral equations for nonlinear Helmholtz equation based on Side's quadrature rules[8], in which the generation of the discrete matrixes is without any singular calculations. Based on the asymptotical compact theory and Stepleman theorem, we prove the stability and the convergence of the nonlinear system. We also solve the nonlinear system by Newton iteration. Since the asymptotic expansions of the errors with the power $O(h^3)$ are shown, the high order $O(h^5)$ can also be achieved using the extrapolation algorithms (EAs)[8,10,17]. Thus, we not only greatly improve the accuracy of the approximation for the equation, but also derive the posterior error estimate which can be used to construct self-adaptive algorithms. Numerical examples support our algorithms and show that the MQMs are fit for practice.

This paper is organized as follows: In Section 2, we construct the MQMs for the nonlinear boundary integral equations and obtain a nonlinear system. In Section 3, we analyze the error about u_h to obtain an asymptotic expansion and a Richardson extrapolation. In Section 4, we discuss Newton iteration extensively by the Ostrowski fixed point theorem. In Section 5, numerical examples show the significance of the algorithm.

2 Mechanical quadrature method

Assume that Γ is a smooth closed curve described by a regular parameter mapping $x(s) = (x_1(s),\ x_2(s)) : [0, 2\pi] \to \Gamma$ that satisfies $|x'(s)|^2 = |x_1'(s)|^2 + |x_2'(s)|^2 > 0$. Let $C^{2m}[0, 2\pi]$ denote the set of $2m$ times differentiable periodic functions with the periodic 2π and $x_i(s) \in C^{2m}[0, 2\pi], i = 1, 2$. Define the following integral operators on $C^{2m}[0, 2\pi]$:

$$\begin{cases} (Ku)(s) = 2\displaystyle\int_0^{2\pi} k(t,s)u(t)\mathrm{d}t, \\ (Hu)(s) = 2\displaystyle\int_0^{2\pi} h(t,s)u(t)\mathrm{d}t, \end{cases} \tag{6}$$

where $u(t) = \tilde{u}(x_1(t), x_2(t))$, $k(t,s) = k^*(x(t), x(s))|x'(t)|$ is a smooth function, and $h(t,s) = h^*(x(t), x(s))x'(t)|$ is a logarithmic weak singular function. Then, Eq. (3) is equivalent to

$$(I - K)u + HG(u) = Hf. \tag{7}$$

In order to obtain the solution, three assumptions[6–7] are given as follows:

(i) $\mathrm{diag}(\Omega) < 1$,

(ii) $g(\cdot, u)$ is measurable for $u \in R$, and $g(x, \cdot)$ is continuous for $x \in \Gamma$,

(iii) $\frac{\partial}{\partial u} g(x,u)$ is Borel measurable and satisfies $0 < l < \frac{\partial}{\partial u} g(x,u) < L < \infty$.

Note that assumption (iii) means that $g(x, u)$ is Lipschitz continuous and strongly monotonous. We shall obtain the uniquely solvability from the assumptions.

Lemma 1[6] *Under these assumptions, if $f \in H^{-1/2}(\Gamma)$, then there is a unique solution in Eq.* (7).

Let $h = 2\pi/n$ ($n \in N$) be the mesh width and $t_j = s_j = jh$ ($j = 0, 1, \cdots, n-1$) be the nodes. Since K is a smooth integral operator with period 2π, we obtain Nyström's approximation with a high accuracy order by the trapezoidal rule[18–19].

$$(K_h u)(s) = h \sum_{j=0}^{n-1} k(t_j, s) u(t_j) \tag{8}$$

with the error estimate

$$(Ku)(s) - (K_h u)(s) = O(h^{2m}). \tag{9}$$

For the logarithmic weakly singular operator H, Nyström's approximation operator can be defined by Sidi's quadrature rules[19],

$$(H_h G(u))(s) = h \sum_{j=0}^{n-1} h_n(t_j, s) G(u(t_j)), \tag{10}$$

where the continuous approximation of the kernel $h_n(t,\tau)$ is defined as

$$h_n(t,s)=\begin{cases} h(t,s), & |t-s|\geqslant h,\\ \dfrac{1}{2\pi}\Big(\log\Big(\dfrac{1}{2\pi}h|x'(s)|\Big)+\varepsilon_\alpha\Big)|x'(s)|, & |t-s|<h \end{cases} \tag{11}$$

with

$$\varepsilon_\alpha=\begin{cases} -\log(2\alpha)-\gamma & \text{for} \quad \alpha>0,\\ 0 \quad \text{for} \quad \alpha=0. \end{cases} \tag{12}$$

The error estimate of H is

$$(HG(u))(s)-(H_hG(u))(s)=2\sum_{\mu=1}^{m-1}\frac{\varsigma'(-2\mu)}{(2\mu)!}(G(u(s)))^{(2\mu)}h^{2\mu+1}+O(h^{2m}), \tag{13}$$

where $\varsigma'(t)$ is the derivative of Riemann zeta function.

Thus, we obtain the numerical approximate equations of Eq. (7),

$$(I-K_h)u_h+H_hG(u_h)=H_hf_h. \tag{14}$$

Obviously, Eq. (14) is a nonlinear equations system with u_h unknowns. When u_h on the boundary Γ are obtained, we can calculate the value $u_h(y)$ for $y\in\Omega$, i.e.,

$$u_h(y)=\frac{h|x'(t_j)|}{2\pi}\sum_{j=1}^{n}\Big[u_h(t_j)k(x(t_j),y)+(G(u(t_j))+f(t_j))h_n(x(t_j),y)\Big], \quad y\in\Omega. \tag{15}$$

3 Asymptotic expansion and extrapolation algorithms

Since K is a continuous operator, according to logarithmic capacity theory[8,20], $(I-K)^{-1}$ is existed and uniformly bounded. Denote $L_h=(I-K_h)^{-1}H_h$ and $L=(I-K)^{-1}H$, then we obtain the following asymptotically compact property.

Theorem 1 *The approximate operator sequence $\{L_h\}$ is an asymptotically compact sequence and convergent to L in $C[0,2\pi]$, i.e.,*

$$L_h\overset{\text{a.c.}}{\to}L.$$

Proof Since $h_n(t,\tau)$ is a continuous approximate of $h(t,\tau)$, the approximate operator $\{H_h\}$ is an asymptotically compact sequence and convergent to H, i,e., $H_h\overset{\text{a.c.}}{\to}H$ in $C[0,2\pi]$ as $n\to\infty$. It implies that for any bounded sequence $\{y_m\in C^{(m)}[0,2\pi]\}$ there exists a convergent subsequence in $\{H_hy_m\}$. Without loss of generality, we assume $H_hy_m\to z$ as $m\to\infty$. From the properties of asymptotically compact convergence and quadrature rules[18–19], we have

$$\begin{aligned}||L_hy_m-(I-K)^{-1}z||&\leqslant||(I-K_h)^{-1}||\cdot||H_hy_m-z||\\ &\quad+||(I-K_h)^{-1}(K_h-K)(I-K)^{-1}z||\to0, \quad m\to\infty,h\to0,\end{aligned}$$

where $||\cdot||$ is the norm of $\pounds(C^{(m)}[0,2\pi],C^{(m)}[0,2\pi])$. It shows that $\{L_h\}$ is an asymptotically compact operator sequence.

Moreover, we show that the sequence L_h is pointwisely convergent to L as $n\to\infty$. In fact, since $H_h\overset{\text{a.c.}}{\to}H$ for all $y\in C^{(m)}[0,2\pi]$, we obtain

$$\|H_hy-Hy\|\to0, \quad h\to0.$$

Following the quadrature rules[19], we derive

$$\|L_h y - Ly\| \leqslant \|(I-K_h)^{-1}\| \cdot ||H_h y - Hy\| + ||(I-K_h)^{-1}(K_h-K)(I-K)^{-1}Hy|| \to 0, \quad h \to 0.$$

Hence, the sequence L_h is pointwisely convergent to L.

Since the sequence L_h is an asymptotically compact sequence and pointwisely convergent to L, the proof of the theorem is completed.

According to the assumptions about the function $G(u)$, we will obtain the conclusion about asymptotically compact convergence[21]

$$\begin{cases} H_h G'(u) \stackrel{\text{a.c.}}{\to} HG'(u), \\ (I-K_h)^{-1}H_h G'(u_h) \stackrel{\text{a.c.}}{\to} (I-K)^{-1}HG'(u). \end{cases}$$

Denote $\overline{K}_h u_h = (I-K_h)^{-1}u_h$, $\overline{H}_h u_h = H_h G(u_h)$, and $\bar{f}_h = H_h f_h$. Then, Eq. (14) is converted into

$$\overline{K}_h u_h + \overline{H}_h u_h = \bar{f}_h. \tag{16}$$

Lemma 2[22] *Let the mapping* $A: R^n \to L(R^n)$, $B: R^n \to L(R^m, R^n)$, *and* $F: R^n \to R^n$ *be continuous and assume that* $||A(x)^{-1}|| < C_1 < \infty$, $||A(x)^{-1}B(x) < C_2 < \infty||$, $||F(x)|| < C_3 < \infty$, $\forall x \in R^n$. *Then, for any* $b \in R^m$, *the equation* $A(x)x = B(x)b + F(x)$ *has a solution, where* $B \in L(R^m, R^n)$ *denotes an* $n \times m$ *matrix or a linear operator.*

Lemma 3[6] *Let* $f \in H^{-1/2}(\Gamma)$ *be given. Then, there exists* $R(f) > 0$ *such that the equations*

$$A_t(u) = u + t(-Ku + HG(u) - Hf) = 0 \tag{17}$$

do not admit any solution for any $t \in [0,1]$ *whenever* $||u||_0 > R(f)$.

Theorem 2 *Under these three assumed conditions and also supposing that* $u, f \in C^3[0, 2\pi]$, *we obtain that*

(i) *there is a solution in Eq.* (16),

(ii) *there is an asymptotic expansion for the solution*

$$u - u_h = \psi(s)h^3 + o(h^3), \tag{18}$$

where $\psi(s) \in C^0[0, 2\pi]$ *is a function independent of* h.

Proof (i) It is obvious that $\overline{K}_h$ and $\overline{H}_h$ are continuous. Since K is a continuous operator, according to the logarithmic capacity theory, $(I-K)^{-1}$ is existed and uniformly bounded. Then $||\overline{K}_h{}^{-1}|| < C_1 < \infty$.

From Lemma 3 and the assumption about $G(u)$, we have

$$||\overline{H}_h u_h|| = h \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |h_{ij} G(u_{jh})| \leqslant h \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |h_{ij}||G(u_{jh})| \leqslant Ch \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |h_{ij}|,$$

where $h_{ij} = h_n(t_i, t_j)$ and $C > 0$. According to the Nyström approximation operator H_h, we have $||\overline{H}_h u_h|| < C_2 < \infty$. So Lemma 2 is satisfied, and there is a solution in Eq. (16).

(ii) From Eq. (7) and Eq. (14), we have

$$(u - u_h) + (Ku - K_h u_h) + (HG(u) - H_h G(u_h)) = (H - H_h)f.$$

Since H is a logarithmic weakly singular operator, from Eq. (13), we have

$$Hf(s) - H_h f(s) = 2\frac{\varsigma'(-2)}{2!} f^{(2)}(s)h^3 + o(h^3).$$

It is equivalent to

$$\begin{aligned}&(u - u_h) + (Ku - K_h u + K_h u - K_h u_h)\\&+ (HG(u) - H_h G(u) + H_h G(u) - H_h G(u_h)) = \varphi_1(s)h^3 + o(h^3),\end{aligned}$$

where $\varphi_1(s) = \varsigma'(-2)f^{(2)}(s)$. Following the error estimates of Eq. (9) and Eq. (13),

$$(u - u_h) + K_h(u - u_h) + H_h(G(u) - G(u_h)) = \varphi(s)h^3 + o(h^3),$$

where $\varphi(s) = \varphi_1(s) + \varsigma'(-2)(G(u(s)))^{(2)}(s)$. Using the mean value theorem of differentials, we obtain

$$(I - K_h + H_h G'(\overline{u}_h))(u - u_h) = \varphi(s)h^3 + o(h^3), \tag{19}$$

where $\overline{u}_h = u_h + t(u - u_h)$ $(0 \leqslant t \leqslant 1)$. Since $(I - K_h)^{-1}$ exists, Eq. (19) can be rewritten as

$$(I - W_h)(u - u_h) = \varphi(s)h^3 + o(h^3),$$

where $W_h = -(I - K_h)^{-1} H_h G'(\overline{u}_h)$ and $W = -(I - K)^{-1} HG'(\overline{u})$.

We have the conclusion $W_h \overset{\text{a.c.}}{\to} W$ and $(I - K_h)^{-1}$ is existed and bounded. So $(I - W_h)^{-1}$ is bounded[8,18], when we set $\psi(s) = (I - W_h)^{-1}\varphi(s)$, the conclusion can be obtained.

Let u_h and $u_{h/2}$ be the solutions of Eq. (14) according to mesh widths h and $h/2$, respectively. From Eq. (18), the h^3-Richardson extrapolation algorithms (EAs)[8,10] can be obtained as

$$u_h^*(s_j) = \frac{1}{7}(8u_{h/2}(s_j) - u_h(s_j)), \quad s_j = jh, \quad j = 0, \cdots, n-1,$$

where the error estimates are $||u_h^*(s_j) - u(s_j)|| = o(h^3)$.

Harnessing the extrapolation result, we can derive the following posteriori error estimate:

$$\begin{aligned}&||u_{h/2}(s_j) - u(s_j)||\\ \leqslant&\Big|\Big|\frac{8}{7}u_{h/2}(s_j) - \frac{1}{7}u_h(s_j) - u(s_j)\Big|\Big| + \frac{1}{7}||u_{h/2}(s_j) - u_h(s_j)||\\ \leqslant&\frac{1}{7}||u_{h/2}(s_j) - u_h(s_j)|| + o(h^3).\end{aligned}$$

Note that the inequality can be used to construct self-adaptive algorithms. Although the extrapolation algorithms are not very complex, they are very efficient to improve the convergent rate.

4 Iterative solution of nonlinear equations

We shall give Newton iteration to solve the nonlinear equations. For convenience, we denote

$$\Psi(z) = (\varphi_1(z), \cdots, \varphi_n(z))^{\mathrm{T}}, \tag{20}$$

where $z = (z_1, \cdots, z_n)^{\mathrm{T}} = u_h$, and

$$\varphi_i(z) = z_i - h\sum_{j=1}^{n} k_{i,j} z_j - h\sum_{j=1}^{n} h_{i,j}(G(z_j) - f_j)$$

with $k_{i,j} = k_n(t_i, t_j)$ and $h_{i,j} = h_n(t_i, t_j)$.

Then, Eq. (14) can be rewritten as

$$\Psi(z) = 0. \tag{21}$$

The Jaccobi matrix of $\Psi(z)$ is

$$A(z) = \Psi'(z) = (\partial_j \varphi_i(z))_{n\times n}. \tag{22}$$

So Newton iteration is constructed

$$z^{k+1} = \omega(z^k), \quad \omega(z) = z - (A(z))^{-1}\Psi(z), \quad k = 0, 1, 2, \cdots. \tag{23}$$

Lemma 4[22] (Ostrowski) *Suppose there is a fixed point $z^* \in \mathrm{int}(D)$ of the mapping: $\omega : D \subset R^n \to R^n$ and the F-derivation of ω at point z^* exists. If the spectral radius of $\omega'(z^*)$ satisfies*

$$\rho(\omega'(z^*)) = \delta < 1. \tag{24}$$

Then, there is an open ball $S = S(z^*, \delta_0) \subset D$ that for $z^0 \in S$, the iterative sequence Eq. (23) is stable and convergent to z^*.

Lemma 5[22] *Suppose $A, C \in L(R^n)$, $||A^{-1}|| < \beta$, $||A - C|| < \alpha$, $\alpha\beta < 1$, then C is invertible and $||C^{-1}|| < \beta/(1 - \alpha\beta)$.*

Theorem 3 *Suppose $\Psi: D \subset R^n \to R^n$ is F-derivative, and z^* satisfies equation $\Psi(z) = 0$. $A : S \subset D \to L(R^n)$ is continuous and invertible at z^*, where S is the neighborhood of z^*. Then, there is a close ball $\overline{S} = \overline{S}(z^*, \delta) \subset S$ that ω is F-derivative at z^*:*

$$\omega'(z^*) = I - (A(z^*))^{-1}\Psi'(z^*). \tag{25}$$

Proof Let $\beta = ||(A(z^*))^{-1}|| > 0$. Since $A(z^*)$ is invertible, and $A(z)$ is continuous at z^*, for $0 < \varepsilon < (2\beta)^{-1}$, $\exists\delta > 0$, when $z \in \overline{S}(z^*, \delta)$, there is $||A(z) - A(z^*)|| < \varepsilon$. According to Lemma 5, $(A(z))^{-1}$ exists and $||(A(z))^{-1}|| \leqslant \beta/(1 - \varepsilon\beta)$ for any $z \in \overline{S}$. So we construct the function

$$\omega(z) = z - (A(z))^{-1}F(z), \ z \in \overline{S}.$$

Since $\Psi(z)$ is derivative at z^*, $\exists\delta > 0$. When $z \in \overline{S}(z^*, \delta)$, we obtain an inequality following the definition of the F-derivation:

$$||\Psi(z) - \Psi(z^*) - \Psi'(z^*)(z - z^*)|| \leqslant \varepsilon||z - z^*||.$$

Consider the derivation of $\omega(z)$

$$\begin{aligned}
&||\omega(z) - \omega(z^*) - [I - (A(z^*))^{-1}\Psi'(z^*)](z - z^*)|| \\
&= || - (A(z))^{-1}\Psi(z) - (A(z^*))^{-1}\Psi'(z^*)(z - z^*)|| \\
&\leqslant ||(A(z))^{-1}(A(z^*) - A(z))(A(z^*))^{-1}\Psi'(z^*)(z - z^*)|| \\
&\quad + ||(A(z))^{-1}(\Psi(z) - \Psi(z^*) - \Psi'(z^*)(z - z^*))|| \\
&\leqslant (2\beta^2\varepsilon||F'(z^*)|| + 2\beta\varepsilon)||z - z^*|| \leqslant c\varepsilon||z - z^*||,
\end{aligned} \tag{26}$$

where $c = 2\beta(\beta||F'(z^*)|| + 1)$. According to the definition about the F-derivation, we obtain the F-derivation of ω at z^*

$$\omega'(z^*) = I - (A(z^*))^{-1}\Psi'(z^*).$$

Using the definition of the matrix A in Eq. (22), we have $\rho(\omega'(z^*)) = 0 < 1$. According to Lemma 4, the iterative sequence is stable and convergent to z^*.

5 Numerical examples

Example 1 Consider Helmholtz equation on a circle plate domain $\Omega = \{(x_1, x_2)|x_1^2+x_2^2 \leqslant 0.5^2\}$ with $\alpha = \sqrt{2}$. We describe the boundary Γ as $(x_1, x_2) = (0.5\cos\theta, 0.5\sin\theta), \theta \in [0, 2\pi]$. The nonlinear boundary condition on Γ is given: $G(u) = u+\sin(u)$ and $f = \mathrm{e}^{0.5\cos\theta+0.5\sin\theta}(\cos\theta + \sin\theta + 1) + \sin(\mathrm{e}^{x_1+x_2})$, then the analytic solution is $\mathrm{e}^{x_1+x_2}$.

In Table 1, we list some errors of the $u_h(y)$ on Γ computed following formulae (14) when set the iteration steps as twenty. We denote the error $e^h(\theta) = |u_h(\theta) - u(\theta)|$, the extrapolation error $\tilde{e}^h(\theta) = |u_h^*(\theta) - u(\theta)|$, the error ratio $r^h(\theta) = e^h(\theta)/e^{h/2}(\theta)$, and the a posteriori error estimate $p^h(\theta) = \frac{1}{7}||u_{h/2}(\theta) - u_h(\theta)||$.

Table 1 Errors $e^h(\theta)$ and $\tilde{e}^h(\theta)$ and error ratio $r^h(\theta)$ when $\theta_1 = \pi/6$ and $\theta_2 = 2\pi/3$ on Γ

n	24	48	96	192	384
$e^h(\theta_1)$	5.32E−4	6.44E−5	8.01E−6	1.01E−6	1.26E−7
$r^h(\theta_1)$		8.26	8.04	8.00	8.00
$\tilde{e}^h(\theta_1)$		2.74E−6	9.89E−8	3.44E−9	1.15E−10
$p^h(\theta_1)$		6.47E−5	8.02E−6	1.01E−6	1.26E−7
$e^h(\theta_2)$	8.61E−4	1.04E−4	1.31E−5	1.64E−6	2.05E−7
$r^h(\theta_2)$		8.27	8.04	8.01	8.00
$\tilde{e}^h(\theta_2)$		4.95E−6	1.77E−7	6.10E−9	2.01E−10
$p^h(\theta_2)$		1.07E−4	1.33E−5	1.65E−6	2.05E−7

From Table 1, we can numerically see $\log_2 r^h(\theta_i) \approx 3$, and $\log_2 \tilde{e}^h(\theta_i)/\tilde{e}^{h/2}(\theta_i) \approx 5$ which agrees with Eq. (18) very well.

Then, since the boundary values on Γ are obtained, we shall obtain u_h at the arbitrary point following Eq. (15).

Example 2[15] For a numerical example, we consider the scattering of a plane wave u by a sound-soft cylinder with a non-convex boomerang-shaped cross section with boundary Γ illustrated in Fig. 1 and described by the parametric representation

$$x(t) = (x_1(t), x_2(t)) = (\cos t + 0.65\cos 2t + 0.65, 1.5\sin t), \quad 0 \leqslant t \leqslant 2\pi.$$

For the scattered wave u, Kress and Sloan[15] solved a exterior Dirichlet problem. Here, we solve a Neumann problem with nonlinear boundary condition. We set $\alpha = \sqrt{2}$, $G(u) = u + \sin(u)$, and $f = \mathrm{e}^{-x_1-x_2}[(-1.5\cos t - \sin t - 1.3\sin 2t)/\sqrt{w} + 1] + \sin(\mathrm{e}^{-x_1-x_2})$ with $w = (1.5\cos t)^2 + (\sin t + 1.3\sin 2t)^2$. Then, the analytic solution is $\mathrm{e}^{-x_1-x_2}$.

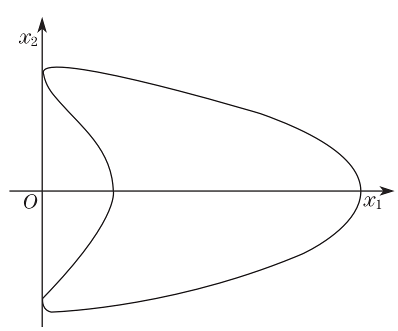

Fig. 1 Boomerang-shaped domain for numerical example

In Table 2, we list some errors of the $u_h(y)$ on Γ computed by formulae (14) and then the u_h at arbitrary point in Ω can be obtained following Eq. (15). We also use the denotes as in Table 1.

Table 2 Errors $e^h(\theta)$ and $\tilde{e}^h(\theta)$ and error ratio $r^h(\theta)$ when $\theta_1 = 0$ and $\theta_2 = \pi/4$ on Γ

n	8	16	32	64	128
$e^h(\theta_1)$	6.13E−3	7.32E−4	8.92E−5	1.10E−5	1.38E−6
$r^h(\theta_1)$		8.37	8.21	8.10	8.00
$\tilde{e}^h(\theta_1)$		7.49E−5	2.71E−6	9.43E−8	3.15E−9
$p^h(\theta_1)$		7.41E−4	8.95E−5	1.11E−5	1.38E−6
$e^h(\theta_2)$	5.41E−3	6.48E−4	7.91E−5	9.77E−6	1.22E−6
$r^h(\theta_2)$		8.35	8.19	8.10	8.00
$\tilde{e}^h(\theta_2)$		6.85E−5	2.47E−6	8.58E−8	2.85E−9
$p^h(\theta_2)$		6.55E−4	7.95E−5	9.78E−6	1.22E−6

6 Conclusions

Generally, there are two main advantages of the MQMs: (i) Evaluating each element of discretization matrices is very simple and straightforward, which does not require any singular integrals. (ii) The algorithm has a high accuracy of $O(h^3)$. According to the numerical results, we find that the larger the scale of the problems is, the more precision the EAs results obtains. In this paper, we only discuss the MQMs for problems with a smooth boundary Γ. It can be generalized to problems with non-smoothed boundary conditions.

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