Math 172 Spring 2012 Worksheet 5 Answers

1.

$$A{\cdot}B = \begin{bmatrix} 4+3 & 6-7 \\ 8-9 & 12+21 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ -1 & 33 \end{bmatrix} \qquad B{\cdot}A = \begin{bmatrix} 4+12 & -4+18 \\ -3+14 & 3+21 \end{bmatrix} = \begin{bmatrix} 16 & 14 \\ 11 & 24 \end{bmatrix}$$

$$A{\cdot}C = \begin{bmatrix} 9-10 \\ 18+30 \end{bmatrix} = \begin{bmatrix} -1 \\ 48 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1-2 & -1-3 \\ 2+6 & -2+9 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}$$

$C \cdot B$ is not possible.

2. A population consists of three age categories: children C_n, mature individuals M_n, and seniors S_n. The population vector B_n is

$$B_n = \begin{bmatrix} C_n \\ M_n \\ S_n \end{bmatrix}$$

The process described below takes place during each step:

25% of the children become mature individuals; 3% of children die

45% of the mature individuals become seniors; 8% of the mature individuals die; each pair of mature individuals produces two children

30% of the seniors die.

a. The transition matrix is

$$A = \begin{bmatrix} 0.72 & 1 & 0 \\ 0.25 & 0.47 & 0 \\ 0 & 0.45 & 0.7 \end{bmatrix}$$

The recursive equation in matrix form is: $B_{n+1} = A \cdot B_n$.

The formula for B_n is: $B_n = A^n \cdot B_0$.

b. Do on calculator:

$$B_3 = A^3 \cdot B_0 = \begin{bmatrix} 175 \\ 68 \\ 137 \end{bmatrix} \qquad B_4 = A^4 \cdot B_0 = \begin{bmatrix} 195 \\ 76 \\ 127 \end{bmatrix}$$

$$B_{20} = A^{20} \cdot B_0 = \begin{bmatrix} 1040 \\ 406 \\ 445 \end{bmatrix} \qquad B_{21} = A^{21} \cdot B_0 = \begin{bmatrix} 1155 \\ 451 \\ 494 \end{bmatrix}$$

c. The total population at $n = 3$ is 380. To find the distribution vector, divide each entry in the vector B_3 by 380:

$$D_3 = \begin{bmatrix} 0.4605 \\ 0.1789 \\ 0.3605 \end{bmatrix}$$

Similarly we get

$$D_4 = \begin{bmatrix} 0.4899 \\ 0.1910 \\ 0.3191 \end{bmatrix} \quad D_{20} = \begin{bmatrix} 0.55 \\ 0.2147 \\ 0.2353 \end{bmatrix} \quad D_{21} = \begin{bmatrix} 0.55 \\ 0.2148 \\ 0.2352 \end{bmatrix}$$

The distribution vector changes from $n = 3$ to $n = 4$, which means that the population has not reached a stable state at $n = 3$.

The distribution vectors at $n = 20$ and $n = 21$ are almost the same, so the population has almost reached a stable state at $n = 20$ (but not quite).

d.

The total population at $n = 20$ is $P_{20} = 1891$ and at $n = 21$ it is $P_{21} = 2100$. In order to explore the issue of exponential behavior, we need to look at the ratio $P_{21}/P_{20} = 1.11$. If this ratio remains the same as we increase n, that would mean that we have exponential behavior with a per-capita growth rate of 11% (so $P_{n+1} = (1+r)P_n$). In order to see whether this happens, we find P_{22}, P_{23} (do this on calculator: find the vectors B_{22}, B_{23} by doing $A^{22} \cdot B_0, A^{23} \cdot B_0$ and then add the three entries in each vector to find P_{22}, P_{23}). We find $P_{22} = 2332, P_{23} = 2591$ so the ratio of consecutive values is: $P_{23}/P_{22} = P_{22}/P_{21} = 1.11$. Since the ratio is constant, we conclude that exponential behavior holds with a per capita growht rate of 11%.