Math 172 Spring Fall 2012 Worksheet 3 Solutions

1. a. $\dfrac{dN}{dt} = 1 - \dfrac{N}{3}$

b. $\frac{dN}{1-\frac{N}{3}} = dt$

to integrate the left hand side let $u = 1 - \dfrac{N}{3}$ so $du = -\dfrac{1}{3}dN$, so $dN = -3du$. The integral becomes $-3\displaystyle\int \frac{du}{u} = -3\ln|u| + C_1 = -3\ln|1 - \dfrac{N}{3}| + C_1$,

and the integral for the right hand side is $t + C_2$. Setting the two integrals equal we get

$$-3\ln|1 - \frac{N}{3}| + C_1 = t + C_2$$

Sovling for N:

$$\ln|1 - \frac{N}{3}| = -\frac{t}{3} + C$$

Exponentiate:

$$|1 - \frac{N}{3}| = e^C \times e^{-\frac{t}{3}}$$

If the initial value is $N(0) = 0$ (this should have been specified in the problem) then the quantity inside the absolute value is positive so we have

$$1 - \frac{N}{3} = e^C \times e^{-\frac{t}{3}}$$

Plug in $t = 0$ to find that $1 = e^C$ (again, assuming $N(0) = 0$) so the equation becomes

$$1 - \frac{N}{3} = e^{-\frac{t}{3}}$$

so $N(t) = 3 - 3e^{-t}3$.

c. $t = 3.3$ (approximately)

to graph the function on the calculator put the calculator back in function mode; use x for t and y for N; set your window to reflect the values you expect to see.

d. there is a stable equilibrium value of 3; this is reflected in the fact that the graph has a horizontal asymptote at $y = 3$ (the values get closer and closer to 3).

2.

$$\frac{dP}{dt} = 0.02P - 12$$

b. $P_{equil} = 12/0.02 = 600$.

c.

$$\frac{dP}{0.02P - 12} = dt$$

to integrate the left hand side set $u = 0.02P - 12$, so $du = 0.02\,dP$, and $dP = \dfrac{du}{0.02}$. The integral of the left hand side becomes

$$\frac{1}{0.02}\int \frac{du}{u} = \frac{1}{0.02}\ln|u| + C_1 = \frac{1}{0.02}\ln|0.02P - 12| + C_1$$

and the integral of the right hand side is $t + C_2$. Set the two integrals equal to each other:

$$\frac{1}{0.02}\ln|0.02P - 12| + C_1 = t + C_2$$

Solve for P:

$$\ln|0.02P - 12| = 0.02t + C$$

Exponentiate:

$$|0.02P - 12| = e^C \times e^{0.02t}$$

Note that the given initial value $P_0 = 700$ makes the quantity inside the absolute value positive, so the equation can be written as

$$0.02P - 12 = e^C e^{0.02t}$$

Plug in $t = 0$ to find the numerical value for e^C: $0.02\times 700 - 12 = e^C$,s o $e^C = 2$ and we obtain $0.02P = 12 + 2e^{0.02t}$. Thus

$$P(t) = \frac{12}{0.02} + \frac{2}{0.02}e^{0.02t} = 600 + 100e^{0.02t}$$

d. The population will reach 800 million at $t = 34.6$ (approximately).

3. a. $\dfrac{dP}{dt} = -0.05P + 8$

b. $P_{equil} = \frac{8}{0.05} = 160$. It is stable equilibrium.

c. The size of the population will approach the equilibrium value regradless of whether is starts above or below.

4. a. $\dfrac{dT}{dt} = k(T - T_s)$.

b. $\dfrac{dT}{dt} = -0.1(T - 20)$

c. Separate the variables:

$$\frac{dT}{T - 20} = -0.1\,dt$$

Integrate each side and set the two integrals equal to each other:

$$\ln|T-20| + C_1 = -0.1t + C_2$$

Solve for T:

$$\ln|T-20| = -0.1t + C$$

Exponentiate:

$$|T-20| = e^C \times e^{-0.1t}$$

Note that the quantity inside the absolute value is positive, since the inital value is $T_0 = 90$, so the equation can be written as

$$T - 20 = e^C \times e^{-0.1t}$$

Plug in $t = 0$ to find the numerical value of e^C: $90-20 = e^C$, so $e^C = 70$ and the equation becomes $T - 20 = 70e^{-0.1t}$, so $T(t) = 20 + 70e^{-0.1t}$.

d. $T(10) = 45.75$.

e. $t = 6.9$ minutes (found from the graph; can also be solved for algebraically).

5. a. $r(0) = 3 > 0$ so increasing

b. set $r(t) = 0$ and solve for t: $-0.8t + 3 = 0$ so $t = 3.75$ years (this is the moment when $r(t)$ changes from positive to negative).

c. separate the variables: $\dfrac{dP}{P} = (-0.8t + 3)\,dt$

integrate each side and set the two integrals equal to each other:

$$\ln(P) + C_1 = -0.8\frac{t^2}{2} + 3t + C_2$$

Solve for P:

$$\ln(P) = -0.4t^2 + 3t + C$$

exponentiate: $P = e^C \times e^{-0.4t^2+3t} = P_0 e^{-0.4t^2+3t}$

d. it will decline to extinciton.

6. A population of salamanders is down to 100 individuals, when the Nature Conservancy begins environmental remediation of their habitat. The per capita growth rate of the population is now (at $t = 0$) at $r = -0.02$ and it is assumed to increase linearly in such a way that at $t = 40$ we will have $r(40) = 0$ (and thereafter r becomes positive).

a. Find the formula for the linear function $r(t)$. The general formula for a linear function is $r(t) = mt + b$ where $b = r(0) = -0.02$ (the y-intercept). To find the value of m plug in $t = 40$; we get $40m - 0.02 = 0$ so $m = 0.02/40 = 0.0005$.

b. separate variables:

$$\frac{dP}{P} = r(t)\,dt = (0.0005t - 0.02)\,dt$$

integrate each side and set the two integrals equal to each other:

$$\ln(P) + C_1 = 0.0005t^2/2 - 0.02t + C_2$$

solve for P:

$$\ln(P) = \frac{2.5}{10^4}t^2 - 0.02t + C$$

exponentiate: $P = e^C \times e^{\frac{2.5}{10^4}t^2 - 0.02t} = P_0 \times e^{\frac{2.5}{10^4}t^2 - 0.02t}$

c. increases without bound