RESEARCH STATEMENT

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Our research has been directed towards three main topics: tight closure and related operations, linkage and linkage-related properties, and homological properties of ideals and modules over commutative Noetherian rings.

1. Tight closure and related operations

1.1. **The localization problem and annihilation of local cohomology.** We recall the definition of tight closure, introduced by Hochster and Huneke in [HH]:

Definition 1.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic $p > 0$. We denote positive powers of p by q, and the set of elements of R which are not contained in any minimal prime ideal by R^o.

a. For any ideal $I \subset R$, $I^{[q]}$ is the ideal generated by the qth powers of elements in I. We say an element $x \in R$ is in the *tight closure*, I^*, of I if there exists a $c \in R^o$, such that $cx^q \in I^{[q]}$ for all large q.

b. We say that the ring R is *weakly F-regular* if every ideal I of R is tightly closed, i.e. $I^* = I$. We say that the ring R is F-regular if every localization of R is weakly F-regular.

The question of whether tight closure commutes with localization has been open since the inception of the theory, until the recent negative answer given by Brenner and Monsky in [BM]. As pointed out in [BM], the following are still open:

- whether the weakly F-regular and F-regular properties are equivalent (known in the Gorenstein case, in the graded case (Smith), and over an uncountable field (Murthy))
- whether tight closure commutes with localization at a single element
- whether tight closure is plus closure for algebras of finite type over a finite field (the plus closure of an ideal I is $I^+ = IR^+ \cap R$, where R^+ is the integral closure of R in an algebraic closure of the fraction field of R).

Our contribution in [V1] was a step toward the equivalence of weak F-regularity and F-regularity for graded rings. This question can be approached via finding uniform annihilators for local cohomology of Frobenius powers. More precisely, Hochster and Huneke have raised the following question:

Question 1. *If* $(R, \mathfrak{m})$ *is a local Noetherian ring of positive characteristic* p *and Krull dimension* d*, and* I *is an ideal primary to a prime* P *with* $\mathrm{ht}(P) = d - 1$*, then does there exist a constant* $b > 0$ *such that*

$$H^0_{\mathfrak{m}}\left(\frac{R}{I^{[q]}}\right) = \frac{I^{[q]} : \mathfrak{m}^\infty}{I^{[q]}}$$

is annihilated by $\mathfrak{m}^{bq}$ *for all* $q = p^e$*?*

An affirmative answer to this question would imply that weak F-regularity is equivalent to F-regularity. In [V1], we provide an affirmative answer for the case when R is a finitely generated positively graded algebra over a field, and I is a homogeneous ideal.

1.2. **$*$-independence, special tight closure, and $*$-spread.** In [V2], we introduce two notions that play an important role in most of our further work: *special tight closure* and $*$*-independence*.

Definition 1.2. Let R be a local Noetherian ring of positive characteristic. We say that a set of elements $\{f_1, \dots, f_n\} \subset R$ is $*$*-independent* if

$$f_i \notin (f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n)^* \qquad \forall\ i = 1, \dots, n.$$

We say that an ideal $I \subset R$ is $*$*-independent* if it has a $*$-independent system of generators.

We note that if R is an analytically irreducible excellent local ring, then the $*$-independent property is independent of a choice of a minimal system of generators of a given ideal I. In other words, if I is $*$-independent, then every minimal system of generators of I is $*$-independent (or, in Epstein's terminology, I is strongly $*$-independent).

The notion of $*$-independence is further pursued by Epstein in [Ep], where he shows that it bears strong connections to a tight closure analogue of the notion of minimal reduction:

Definition 1.3. Let $J \subset I$ be ideals in a Noetherian ring of positive characteristic.

We say that J is a $*$*-reduction* of I if we have $I \subset J^*$. We say that J is a *minimal* $*$*-reduction* of I if it is minimal (with respect to inclusion) among the $*$-reductions of I.

Epstein shows that if $J \subseteq I$ is a $*$-reduction of I, then J is a minimal $*$-reduction of I if and only if it is $*$-independent. Epstein goes on to prove that all the minimal $*$-reductions of a given ideal I have the same number of generators, which he calls the $*$-spread of I (by analogy with the analytic spread). A key ingredient in Epstein's proof is the notion of special tight closure, introduced in [V2]. In joint work with Epstein, [EV], we pursue the notion of $*$-spread and provide an asymptotic formula for $*$-spread in terms of length, without reference to $*$-reductions.

The definition of special tight closure is given below:

Definition 1.4. Let $(R, \mathfrak{m})$ be a Noetherian local ring of positive characteristic p. We say that x is in the *special tight closure of an ideal* I, I^{*sp}, if there exists a fixed $q_0 = p^{e_0}$ and $c \in R$ such that $cx^q \in \mathfrak{m}^{q/q_0} I^{[q]}$ for all $q = p^e$ (equivalently, $x^{q_1} \in (\mathfrak{m} I^{[q_1]})^*$ for some $q_1 = p^{e_1}$).

The definition of special tight closure is a first instance of a version of tight closure involving coefficients. Later instances are the $\mathfrak{a}$-tight closure of Hara and Yoshida ([HY]), and the author's $\mathfrak{a}$-closure in [V6] (which will be discussed later in detail).

We consider special tight closure to be our main contribution to tight closure theory. Our results with Huneke on the special tight closure decomposition ([HV1]) greatly generalize earlier results of K. E. Smith([S1]) in the graded case. Roughly speaking, the special tight closure (which would perhaps be more aptly named the special part of tight closure, since it is not a closure operation in its own right) consists of those elements of the tight closure which are more deeply embedded in the ring than the generators of the original ideal. In a graded set-up, this amounts to having larger degrees.

The main result of [HV1] (an earlier, weaker version of which has been obtained in [V2]) states that, with certain assumptions on the ring, the tight closure of any ideal can be obtained from the ideal itself, plus the special tight closure.

Theorem 1.5 (Theorem 2.1, [HV1])**.** *Let* $(R, \mathfrak{m})$ *be an excellent normal ring of positive characteristic, with perfect residue field. Then we have* $I^* = I^{*sp} + I$.

The assumptions that the ring is normal and has perfect residue field cannot be removed from the statement above. However, Epstein proves the following more general version in [Ep]. Here, the notation $J^{1/q}$, where J is an ideal of R, stands for the ideal of elements of R whose qth powers belong to J.

Theorem 1.6 ([Ep])**.** *Let $(R, \mathfrak{m}, k)$ be an excellent analytically irreducible local domain of characteristic $p > 0$. Assume that the normalization $\overline{R}$ of R has the same residue field k as R. Then for every ideal $I \subset R$, there exists $q' = p^{e'}$ such that*

$$I^* = (I^{[q']} + (I^{[q']})^{*sp})^{1/q}.$$

1.3. **Applications of special tight closure.** Special tight closure has lead us to the main results in [V5], [EV], and [FVV], and has played an important role in [VV].

We now expand on those results which we view as applications of special tight closure.

In [V5], our goal is to prove tight closure analogues of results of Watanabe about chains and families of integrally closed ideals. The main results are:

Theorem 1.7. *Let $(R, \mathfrak{m})$ be a Noetherian local excellent normal ring with perfect residue field, and let $J \subset I$ be tightly closed ideals.*

Let $\mathcal{F}(J, I)$ be the set of ideals I' such that $J \subseteq I' \subseteq I$, $\lambda(I/I') = 1$, and I' is tightly closed.

Then $\mathcal{F}(J, I)$ is non-empty. In particular, if I and J are $\mathfrak{m}$-primary, here exists a sequence $J = I_0 \subset I_1 \subset \ldots \subset I_n = I$ consisting of tightly closed ideals, with $\lambda(I_{i+1}/I_i) = 1$ for all $i = 1, \ldots, n-1$.

In the following result, l denotes the $*$-spread of I with respect to J, i.e. the minimal number of generators for an ideal $K/J \subset R/J$, with $J \subset K \subset I$, such that K is a minimal $*$-reduction of I.

Theorem 1.8. *Let R, J, I be as above.*

The ideals in $\mathcal{F}(J, I)$ are in one-to-one correspondence with points on the grassmanian variety of $(l-1)$-dimensional subspaces in the l-dimensional vector space $V = I/(J + I^{*sp})$.

Along the way, we obtain a useful characterization of minimal $*$-reductions in terms of special tight closure.

Theorem 1.9. *Let $(R, \mathfrak{m})$ be a Noetherian local excellent normal ring with perfect residue field, and let I be a tightly closed ideal. Then $K \subset I$ is a minimal $*$-reduction of I if and only if the images of a minimal set of generators for K form a basis in I/I^{*sp}.*

This result is used in ongoing work (with Fouli and Vassilev) on finding a formula for the *-core of an ideal. By analogy with the corresponding notion for integral closure, we have the following definition:

Definition 1.10. Let R be a Noetherian ring of positive characteristic, and $I \subset R$ an ideal. The $*$-*core* of I is defined to be the intersection of all the minimal $*$-reductions of I.

Fouli and Vassilev prove that the $*$-core and core of an ideal I coincide when the $*$-spread and the spread coincide (such is the case when I is contained in the tight closure of an ideal generated by a system of parameters, and the ring has infinite residue field). In general, the $*$-core contains the core, since there are more reductions that $*$-reductions.

The main result we obtain in [FVV] is given below.

Theorem 1.11. *Let R be a Noetherian local excellent normal ring with perfect residue field.*

Let $J = (f_1, \ldots, f_n) \subseteq I = J + (u_1, \ldots, u_s)$, with $u_1, \ldots, u_s \in J^{*sp}$.

a. Assume that $(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n) : f_i \subseteq J : I$ for all $i \in \{1, \ldots, n\}$, and $u_j(J : u_j) \subseteq J(J : I)$ for all $j \in \{1, \ldots, s\}$. Then $* - \mathrm{core}(I) \subseteq J(J : I)$.

b. Assume that $u_j(J : I) \subseteq \mathfrak{m}J(J : I)$ for all $j \in \{1, \ldots, s\}$. Then $* - \mathrm{core}(I) \supseteq J(J : I)$.

The assumptions of both a. and b. above hold in the following two cases:

- R is a Cohen-Macaulay ring, and $J = (x_1^t, \ldots, x_d^t)$, where $x_1, \ldots, x_d$ is a system of parameters with $x_1, x_2 \in \tau$ (τ denotes the test ideal of R), and $t \geq 3$;
- R has $\mathfrak{m}$-primary test ideal, $J = J_0^{[q]}$, $I = I_0^{[q]}$ for some ideals $J_0 \subset I_0$ with J_0 a minimal $*$-reduction of I_0, and q sufficiently large.

Thus, in both of these situations we have $* - \mathrm{core}(I) = J(J : I)$.

Special tight closure has also played a major role in joint work with Vassilev in [VV], where we determine the only normal rings with perfect residue field for which $I^* = I : \tau$ holds for all ideals $I \subset R$ (where τ denotes the test ideal of R) are the weakly F-regular rings (for which $I^* = I$, and $\tau = R$). Further discussion of this result is deferred for the next subsection, which deals with the test ideal in more detail.

1.4. **The test ideal.** The notion of test ideal plays an important role in the theory of tight closure. It is the basis of the strong connection between tight closure and birational geometry. The work of Smith ([S2]) and Hara ([Ha]) establishes the correspondence between test ideals and multiplier ideals that arise in vanishing theorems. More recently, a new version of tight closure has been introduced in [HY], giving rise to a notion of generalized test ideals which correspond to the multiplier ideals of pairs. The definition of the "classical" test ideal is given below.

Definition 1.12. Let R be a Noetherian ring of positive characteristic. The *test ideal* of R is the ideal

$$\tau_R = \bigcap (I :_R I^*),$$

where the intersection runs over all ideals $I \subset R$.

Huneke introduced the notion of strong test ideal in [Hu], with the goal of bounding the degrees of the equations of integral dependence satisfied by the elements of I^* (which are therefore also in $\overline{I}$) over I.

Definition 1.13. An ideal $T \subset R$ is called a *strong test ideal* if $TI^* = TI$ for all ideals $I \subset R$.

The number of generators of such a strong test ideal provides a bound for the degrees of the above-mentioned equations. The main result of [V3] is the following:

Theorem 1.14. *Let $(R, \mathfrak{m})$ be a Noetherian local reduced ring of positive characteristic, in which the test ideal commutes with completion (for instance, if R is $\mathbf{Q}$-Gorenstein, or the localization of an $\mathbf{N}$-graded ring at the maximal homogeneous ideal). Then the test ideal τ_R is a strong test ideal.*

A generalization of this result is pursued by Enescu in [En].

The proof of Theorem 1.14 actually yields the stronger statement that $I : \tau$ is tightly closed for every ideal $I \subset R$. Thus, we observe that the number of generators of the test ideal provides an upper bound for two numerical invariants defined in [V5]:

$t(R) := \inf\{k \mid$ there exists a sequence of tightly closed ideals of type k

that are cofinal with the powers of $\mathfrak{m}\}$, and

$c(R) := \inf\{k \mid \ (I^*)^k \subseteq I$ for every ideal $I \subseteq R\}$

The relationship between these two invariants is studied in [V5].

Theorem 1.14 also constitutes one of the main motivations for our joint work with Vassilev in [VV]. Starting with the observation that $I^* \subseteq \tau I : \tau$ whenever the assumptions in Theorem 1.14 hold, we ask when does equality take place? In [V4], it was observed that the equality takes place for every ideal of finite projective dimension in a Gorenstein ring. The equality also holds for every ideal $I \subset R$, if R is a one-dimensional complete local ring with infinite residue field. However, if we assume that R is normal Cohen-Macaulay with perfect residue field, the main result in [VV] shows that the equality $I^* = \tau I : \tau$ *cannot* hold for every ideal $I \subset R$ unless R is weakly F-regular.

1.5. **Variations of tight closure.** In [HY], Hara and Yoshida introduce the $\mathfrak{a}$-tight closure, an operation which generalizes the usual tight closure (which is obtained for $\mathfrak{a} = R$). Their definition is as follows:

Definition 1.15 ([HY])**.** Let R be a Noetherian ring of positive characteristic p, and let $\mathfrak{a}, I \subset R$ be ideals. We say that $x \in I^{*\mathfrak{a}}$, the $\mathfrak{a}$-*tight closure* of I, if there exists a $c \in R^o$ such that $cx^q\mathfrak{a}^q \subseteq I^{[q]}$ for all $q = p^e$.

Their main motivation is the fact that the generalized test ideals obtained from these operations, $\tau(\mathfrak{a}) = \bigcap_{I\subset R}(I :_R I^{*\mathfrak{a}})$ correspond in a geometric set-up to multiplier ideals for pairs, thus generalizing the results of [S2] and [Ha].

One drawback of the Hara-Yoshida $\mathfrak{a}$-tight closure is the fact that it is not a closure operation, in the sense that $(I^{*\mathfrak{a}})^{*\mathfrak{a}} \neq I^{*\mathfrak{a}}$. This can be seen for example when $(R, \mathfrak{m})$ is local, I is $\mathfrak{m}$-primary, and $\mathfrak{a} = (f)$ is a principal ideal, with $f \in \mathfrak{m}\setminus I^*$.

In [V6], we introduce a new version of $\mathfrak{a}$-tight closure, which we shall henceforth refer to as $\mathfrak{a}$-closure.

Definition 1.16 ([V6])**.** Let R be a Noetherian ring of positive characteristic p, and let $\mathfrak{a}, I \subset R$ be ideals. We say that $x \in {}^{\mathfrak{a}}I^*$, the $\mathfrak{a}$-*closure* of I, if there exists a $c \in R^o$ such that $cx^q\mathfrak{a}^q \subseteq \mathfrak{a}^q I^{[q]}$ for all $q = p^e$.

This definition incorporates both the idea of Hara and Yoshida of allowing a power of an ideal to play the role of multiplier, and the idea of considering coefficients in a high power of some ideal for the generators of $I^{[q]}$ (which was previously used in the definition of the special tight closure).

There are two advantages to $\mathfrak{a}$-closure: first, it is a true closure operation (${}^{\mathfrak{a}}({}^{\mathfrak{a}}I^*)^* = {}^{\mathfrak{a}}I^*$ for every ideal I); second, there is a multiplicity that allows us to determine membership in the $\mathfrak{a}$-closure, in the case when both $\mathfrak{a}$ and I are $\mathfrak{m}$-primary. This phenomenon is analogous to the way in which membership in the integral closure of an ideal can be determined using the Hilbert-Samuel multiplicity, and membership in the tight closure can be determined using the Hilbert-Kunz multiplicity.

We call this new multiplicity the *joint Hilbert-Kunz multiplicity* of the pair $(\mathfrak{a}, I)$.

Definition 1.17 ([V6])**.** Let $\mathfrak{a}, I \subset R$ be $\mathfrak{m}$-primary ideals in a Noetherian local ring of positive characteristic p. The *joint Hilbert-Kunz* multiplicity of the pair $(\mathfrak{a}, I)$ is

$$e_{HK}(\mathfrak{a}, I) = \lim_{q\to\infty} \frac{1}{q^d}\lambda\left(\frac{R}{\mathfrak{a}^q I^{[q]}}\right),$$

where d is the Krull dimension of R, and q denotes a power of p.

Similar to the Hilbert-Samuel and Hilbert-Kunz multiplicities, this multiplicity plays the role of leading coefficient for the length function appearing in the definition:

Theorem 1.18 ([V6]). *Let $(R, \mathfrak{m})$ be a local Noetherian ring of positive characteristic and Krull dimension d, and let $\mathfrak{a}, I \subset R$ be $\mathfrak{m}$-primary ideals. Then*

$$\lambda\left(\frac{R}{\mathfrak{a}^q I^{[q]}}\right) = e_{HK}(\mathfrak{a}, I)q^d + \mathcal{O}(q^{d-1}).$$

As mentioned above, this multiplicity can be used to test for membership in $\mathfrak{a}$-closure:

Proposition 1.19 ([V6]). *Assume that R is analytically unramified and formally equidimensional, and has test elements for the usual tight closure. Let $\mathfrak{a}, I, J \subset R$ be $\mathfrak{m}$-primary ideals, with $I \subseteq J$. Then $J \subseteq {}^{\mathfrak{a}}I^*$ if and only if $e_{HK}(\mathfrak{a}, I) = e_{HK}(\mathfrak{a}, J)$.*

One of the main results in [V6] connects the $\mathfrak{a}$-tight closure with the Hara-Yoshida $\mathfrak{a}$-tight closure, at the level of test ideals:

Theorem 1.20 ([V6]). *Let R be a Gorenstein finitely generated graded algebra over a field of positive characteristic. Assume that R has Krull dimension at least two, and let $\mathfrak{a}$ be a homogeneous R^+-primary ideal. Then we have $\tau(\mathfrak{a}) = T_{\mathfrak{a}}$, where*

$$\tau(\mathfrak{a}) = \bigcap_{I \subset R} (I : I^{*\mathfrak{a}}), \qquad \text{and} \qquad T_{\mathfrak{a}} = \bigcap_{I \subset R} (I : {}^{\mathfrak{a}}I^*).$$

Both variants of $\mathfrak{a}$-tight closure, as well as the joint Hilbert-Kunz multiplicity, have a version in which the power $\mathfrak{a}^q$ appearing in the definition is replaces by $\mathfrak{a}^{\lceil tq \rceil}$, where $t > 0$ is a fixed real number. We will refer to the resulting operations as $I^{*\mathfrak{a}^t}$ for the Hara-Yoshida version and ${}^{\mathfrak{a}^t}I^*$ for the new version of [V6], and we will denote the resulting multiplicity $e_{HK}(\mathfrak{a}^t, I)$. The study of how these notions depend on t is started in [V6] and continued in [V7]. In [V6], we prove that $e_{HK}(\mathfrak{a}^t, I)$ is continuous as a function of t, and for sufficiently small values of t it can be expressed in terms of the Hilbert-Kunz multiplicity of I and the Hilbert-Samuel multiplicity of $\mathfrak{a}$. More precisely, we have

$$e_{HK}(\mathfrak{a}^t, I) \leq e_{HK}(I) + \frac{\ell e(\mathfrak{a}) t^d}{d!}$$

for all $t > 0$, and if R is excellent and analytically irreducible, then there exists a $t_0 > 0$ such that equality holds for all $t \leq t_0$. Here ℓ denotes the $*$-spread of I, and d is the Krull dimension of R.

We also address the following question:

Question 2. *Given ideals $\mathfrak{a}, I \subset R$, and a fixed $t_0 \geq 0$, does there exist an $\epsilon > 0$ such that $I^{*\mathfrak{a}^t} = I^{*\mathfrak{a}^{t_0}}$, and ${}^{\mathfrak{a}^t}I^* = {}^{\mathfrak{a}^{t_0}}I^*$ for all $t \in [t_0, t_0 + \epsilon]$?*

A positive answer implies that $I^{*\mathfrak{a}^t}$, ${}^{\mathfrak{a}^t}I^*$ are constant on intervals of the form $[t_0, t_1)$. The end-points of such an interval are called jumping numbers, by analogy with the corresponding notions for multiplier ideals and generalized test ideals. In [V6], we give positive answers to this question in several cases, and show that this question is also related to the existence of test exponents for the usual tight closure.

In a different direction, we ask what happens to $I^{*\mathfrak{a}^t}$, ${}^{\mathfrak{a}^t}I^*$ when $t \to \infty$? For the Hara-Yoshida version, the answer is trivial when I is $\mathfrak{m}$-primary, since we have $\mathfrak{a}^{tq} \subseteq I^{[q]}$ if $t \gg 0$, and thus $I^{*\mathfrak{a}^t} = R$ for all $t \gg 0$. The situation for the $\mathfrak{a}$-closure turns out to be much more subtle.

We will denote the union of all ${}^{\mathfrak{a}^t}I^*$ when $\mathfrak{a}, I$ are fixed and $t > 0$ varies by ${}^{\mathfrak{a}^\infty}I^*$. It is not hard to see that we always have ${}^{\mathfrak{a}^\infty}I^* \subseteq \overline{I}$, where $\overline{I}$ denotes the integral closure of I. In [V7], we see that equality holds if $\mathfrak{a}, I$ are projectively equivalent, i.e. if there exist positive integers k, l such that $\overline{\mathfrak{a}^k} = \overline{I^l}$.

The general question we address in [V7] is:

Question 3. *If $\mathfrak{a}, I \subset R$ are $\mathfrak{m}$-primary ideals, and $x \notin I$, what can be said about the behavior of $D(\mathfrak{a}^t; I; x) := e_{HK}(\mathfrak{a}^t, I) - e_{HK}(\mathfrak{a}^t, (I, x))$ as a function of t?*

$D(\mathfrak{a}^t; I; x)$ can be thought of as a measure of the failure of x to belong to ${}^{\mathfrak{a}^t}I^*$. In particular, $x \in {}^{\mathfrak{a}^\infty}I^*$ if and only if $D(\mathfrak{a}^t; I; x) = 0$ for $t \gg 0$.

Quite generally, it turns out that $D(\mathfrak{a}^t; I; x)$ is bounded by Ct^{d-1}, where d is the Krull dimension of the ring, and $C > 0$ is a constant. We define the *level ideals* of I with respect to $\mathfrak{a}$ to be

$$I_{\mathfrak{a},j} := \{x \in R \mid \exists C > 0 \text{ such that } D(\mathfrak{a}^t; I; x) \leq Ct^{d-1-j}\}.$$

We prove that $I_{\mathfrak{a},j}$ are indeed ideals, and the assignment $I \to I_{\mathfrak{a},j}$ is a closure operation.

Our main results in [V7] give an algorithm for computing $D(\mathfrak{a}^t; I; x)$ when $t \gg 0$, and $\mathfrak{a}, I \subset P := k[x_1, \ldots, x_d]$ are monomial ideals in a polynomial ring. This algorithm relies on the geometry of the Newton polyhedron of $\mathfrak{a}$. As a corollary, we obtain a characterization of when two monomial ideals $\mathfrak{a}, \mathfrak{b}$ determine the same level ideals, i.e. when $I_{\mathfrak{a},j} = I_{\mathfrak{b},j}$ for all ideals $I \subset P$, and for all $j = 1, \ldots, d$. This characterization depends on the Rees valuations associated to $\mathfrak{a}, \mathfrak{b}$, and the geometry of the corresponding Newton polyhedra.

2. Linkage and related properties

Linkage is a relationship between ideals in a Gorenstein ring. We say that I and J are *linked* (or *directly linked*) if there exists an ideal $(\underline{x}) \subset I \cap J$ generated by part of a system of parameters, such that $I = (\underline{x}) : J$, and $J = (\underline{x}) : I$ (the second condition follows automatically from the first under the additional assumption that J is unmixed). As seen in [PS2], ideals that are linked share many important properties.

Linkage has been used as an important tool in many of our results. The following are illustrations of our earlier results that use linkage:

Proposition 2.1 ([V3])**.** *Let $(R, \mathfrak{m})$ be a Gorenstein local ring, and let $\underline{x}$ be a system of parameters. Assume that $J \subset R$ is an $\mathfrak{m}$-primary ideal with the following properties:*

- $(\underline{x}) \subseteq \mathfrak{m}J$
- *$K = (\underline{x}) : J$ is tightly closed.*

Then J is a strong test ideal, i.e. $JI^* = JI$ for every ideal $I \subset R$.

This result can be read as saying that the property of being tightly closed is in some sense dual to the property of being a strong test ideal. Another property that is dual to being tightly closed is seen in the following:

Proposition 2.2 ([V5])**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, let $\mathfrak{a}$ be an $\mathfrak{m}$-primary irreducible ideal, and $J \supseteq \mathfrak{a}$ a tightly closed ideal. Write $\mathfrak{a} : J = \mathfrak{a} + (f_1, \ldots, f_n)$. Then the ideal $K = (\mathfrak{a}, f_1, \ldots, f_{n-1}) : f_n$ is big, i.e. every ideal containing K is tightly closed.*

2.1. Tight closure and linkage classes.

The connections between tight closure and linkage in Gorenstein rings are explored in more depth in [V4]. This investigation is motivated by the question of the relationship between the tight closure I^* and the ideal $I : \tau$, where τ is the test ideal of T. Since the latter ideal is in general larger, we seek to characterize the elements that multiply $I : \tau$ into I^*. The main result of [V4] states that if R is a Gorenstein ring, and I is an unmixed ideal, then $(\tilde{I})^*(I : \tau) \subseteq I^*$, where $\tilde{I}$ stands for the sum of all the ideals in the linkage class of I (i.e. all ideals that can be obtained as iterated links of I).

2.2. The almost Gorenstein property.

A different aspect of the role of linkage in our work is seen in [HV2]. In joint work with Huneke, we introduce classes of rings that are close to being Gorenstein. We take the defining property of Gorenstein rings to be the fact that they are Cohen-Macaulay, and for every system of parameters $\underline{x}$, and every ideal $I \supseteq (\underline{x})$, we have $(\underline{x}) : ((\underline{x}) : I) = I$. In view of the fact that $R/(\underline{x})$ is its own injective hull, this is an incarnation of Matlis duality. The sense in which the rings studied in [HV2] are close to being Gorenstein is that they are Cohen-Macaulay, and for every system of parameters $\underline{x}$, and every ideal $I \supseteq (\underline{x})$, we have $(\underline{x}) : ((\underline{x}) : I) \subseteq I : \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal. We prove that rings with this property arise naturally as specializations of rings of countable Cohen-Macaulay type.

A stronger property that could also be termed "almost Gorenstein" deals with a property of the canonical module ω_R. Namely, we require that for every $x \in \mathfrak{m}_R$, there exists a linear functional $f : \omega_R \to R$ such that $x \in \mathrm{Im}(f)$. Rings that can be written as the quotient of an Artinian Gorenstein ring by the socle element satisfy this property. These rings were studied by Teter in [Te], and we term them *Teter rings*. Teter proved an implicit characterization of such rings (without reference to an outside Gorenstein rings) in terms of the existence of an isomorphism between $\mathfrak{m}$ and $\mathfrak{m}^{\vee} = \mathrm{Hom}_R(\mathfrak{m}, E_R(R/\mathfrak{m}))$, with an additional technical condition. We strengthen Teter's result by showing that the above-mentioned technical condition is superfluous. An extension of our result is pursued by Ananthnarayanin [An], where he gives a characterization of rings that can be obtained as quotients of Artinian Gorenstein rings by ideals of colength two.

We give a complete description of the Artinian rings with the almost Gorenstein property that can be written as quotients of a polynomial ring by a monomial ideal of type at most three, and we prove that in this case the two versions of the almost Gorenstein property are equivalent (but this equivalence does not hold in general).

The homological properties of rings with the almost Gorenstein properties are considered in on-going joint work with Striuli discussed in Section 3.3 below.

3. Homological algebra

The homological component of our work is contained in [HSV], [KV], and [SV].

3.1. Vanishing of Ext and Tor.

In [HSV] (joint work with Huneke and Şega), we discuss the vanishing of cohomology of finitely generated modules over Cohen-Macaulay local rings $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0$. This work is motivated in part by the Auslander-Reiten conjecture, which states that if $(R, \mathfrak{m})$ is a commutative Noetherian local ring, and M a finitely generated module with $\mathrm{Ext}^i_R(M, M) = 0$, and $\mathrm{Ext}^i_R(M, R) = 0$ for all $i > 0$, then M is free. Assuming that $\mathfrak{m}^3 = 0$, we prove that the conjecture is true, and moreover it suffices to have $\mathrm{Ext}^i_R(M, M \oplus R) = 0$

for any four consecutive values of $i > 2$ (if, in addition, R is Gorenstein, a single value of $i > 0$ suffices). A more general assumption is obtained if the assumption $\mathfrak{m}^3 = 0$ is replaced by $\mathfrak{m}^2 M = 0$.

Based on our examination of the vanishing of Tor and Ext, we propose the following conjecture:

Question 4. *Let $(R, \mathfrak{m})$ be an Artinian local ring, and M, N finitely generated R-modules with $\mathfrak{m}^2 M = \mathfrak{m}^2 N = 0$. If $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$, does it follow that $\mathfrak{m}^3 = 0$?*

The vanishing assumption above imposes strong restrictions on the Poincaré series of the reside field of R; based on these restrictions, we see that the conjecture holds for many classes of rings (complete intersections of codimension at least three, Golod rings, Koszul rings, rings with irrational Poincaré series, etc.) We also prove that the conjecture holds for standard graded rings.

3.2. **Finite projective dimension and socle degrees of Frobenius powers.** A different homological direction of our work (joint with Kustin, in [KV]) is centered on determining whether a homogeneous ideal J in a Gorenstein graded ring has finite projective dimension, based on numerical properties relating the socle degrees of J and of its Frobenius powers, $J^{[q]}$. The idea of giving conditions for J to have finite projective dimension in term of the behavior of $J^{[q]}$, or more generally the behavior of J with respect to the Frobenius endomorphism, has been previously considered in work of Herzog, Avramov-Miller, etc. (where the finite projective dimension is shown to follow from the vanishing of certain Tor modules), and Kunz (where it is shown that the regularity of the ring is equivalent to a certain behavior of the Hilbert-Kunz function of the maximal ideal).

It was previously known that if J has finite projective dimensions, then the socles $(J : \mathfrak{m})/J$ and $(J^{[q]} : \mathfrak{m})/J^{[q]}$ have the same dimension, that if the generators of $(J : m)/J$ have degrees $d_1, ..., d_l$, then the degrees of the generators of $(J^{[q]} : \mathfrak{m})/J^{[q]}$ are given by $qd_i - (q-1)a(R)$, where $a(R)$ is the a-invariant of R. This is based on the fact that the socle degrees can be read off from the tail of the resolution, and the resolution of $R/J^{[q]}$ can be obtained by raising the resolution of R/J to the qth power (by [PS1]).

Our main result in [KV] is the fact that if we assume R to be a complete intersection, then the converse of the above-mentioned result holds (i.e. the relationship between socle degrees of J and $J^{[q]}$ stated above implies that J has finite projective dimension). The complete intersection hypothesis is used in a technical calculation of certain Tor modules, and also in order to apply the result of Avramov and Miller in [AM], which shows that the vanishing of certain Tor modules implies finite projective dimension. We do not know whether our theorem still holds when the complete intersection hypothesis is replaced by Gorenstein, but give partial results in the Gorenstein F-pure case.

A different direction in which our work in [KV] has been extended is by considering numerical relationships between the degrees of the socle generators for J and for $J^{[q]}$ that ensure that the resolutions of the two ideals are related (the case of finite projective dimension is a particular case of this, since in this case the resolution of $J^{[q]}$ is obtained by simply raising all the entries in matrices appearing in the resolution of J to the qth power). This question has been pursued in work of Kustin-Ulrich, and Kustin-Rahmati, where partial results in this direction are obtained.

3.3. **The resolution of the canonical module of almost Gorenstein rings and totally reflexive modules.** In on-going work with Striuli, we focus on homological properties of rings with the almost Gorenstein property (see Section 2.2 above for the definition of almost Gorenstein). The homological property we focus on is the existence of non-free totally reflexive modules. We recall the definition:

Definition 3.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring. An R-module G is *totally reflexive* if there exists an exact complex of finitely generated free R-modules

$$\mathbf{F}: \quad \ldots \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \ldots$$

such that G is the cokernel of d_0, and $\operatorname{Hom}(\mathbf{F}, R)$ is exact.

The totally reflexive modules serve as building blocks for Gorenstein homological algebra, similar to the way in which the free modules are the building blocks for the usual homological algebra. When the ring R is Gorenstein, the totally reflexive R-modules are exactly the maximal Cohen-Macaulay modules.

The question we would like to answer is:

Question 5. *If R is an almost Gorenstein ring, does R admit any non-free totally reflexive modules?*

This is a step toward a more ambitious project, namely:

Question 6. 1. *Give necessary and sufficient conditions in certain families of rings for non-free totally reflexive modules to exist.*

2. *If we know that non-free totally reflexive modules exist, find a construction to build one, and if the ring is not Gorenstein, to build infinitely many.*

The latter question is prompted by the results of [CPST], where the following dichotomy is shown:

Theorem 3.2 ([CPST])**.** *Let R be a commutative Noetherian ring. Assume that the set of isomorphism classes of indecomposable totally reflexive R-modules is finite. Then either this set has exactly one element, represented by R, or R is Gorenstein (and an isolated singularity, or, if R is complete, even a simple hypersurface singularity).*

Our approach to the problem of showing that a ring R does not admit non-free totally reflexive modules is via studying the resolution of the canonical module ω_R. We are able to prove, for a certain class of Artinian almost Gorenstein rings (which includes the Teter rings as a particular case), that a copy of the residue field splits out of the second syzygy of ω_R. Consequently, such rings do not admit any non-free totally reflexive modules (since one of the requirements for a module to be totally reflexive is that $\mathrm{Tor}_i(M, \omega_R) = 0$ for all $i > 0$).

On the other hand, there are examples of almost Gorenstein rings that admit non-free totally reflexive modules. These are provided by a result in [HV2], where we show that any quotient of a ring of finite (or countable) Cohen-Macaulay type by a "sufficiently general" system of parameters is almost Gorenstein. If the sufficiently general system of parameters can be chosen inside the square of the maximal ideal, so that the resulting quotient ring is an embedded deformation, then a result of [AGP] asserting that such rings have non-free totally reflexive modules can be applied. We are able to explicitly construct such an example.

A different class of rings that we attempt to study is the class of rings of small Gorenstein colength (in the sense of [An], the Gorenstein colength of an Artinian ring R is the smallest value of $\lambda(S) - \lambda(R)$, where S is a Gorenstein ring mapping onto R). We point out that the rings of Gorenstein co-length one are exactly the Teter rings. We have found examples of rings of Gorenstein colength two that admit non-free totally reflexive modules. These examples arise as embedded deformations. We ask:

Question 7. *Let R be an Artinian ring of Gorenstein colength two. If R admits non-free totally reflexive modules, is R isomorphic to $S[X]/(X^2)$, for some Gorenstein ring S?*

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