Math 172 Fall 2012 Handout 4 September 6

Solving affine equations by separation of variables and integration

We wish to solve a differential equation of the type

$$\frac{dP}{dt} = rP - c \qquad \text{or } \frac{dP}{dt} = c - rP$$

First we separate the variables. We obtain:

$$\frac{dP}{rP - c} = dt \qquad \text{or } \frac{dP}{c - rP} = dt \tag{1}$$

In order to integrate, recall

$$\int \frac{dx}{x} = \ln(x) + C$$

When we have an expression in the denominator instead of x, we have to use the u-substitution method for finding the integral.

Let's concentrate of the first version of the equation, $\frac{dP}{dt} = rP - c$. Set $u = rP - c$. Then $du = r\,dP$. This means that we have to replace dP by $\frac{du}{r}$ so we have:

$$\int \frac{dP}{rP - c} = \int \frac{du}{ru} = \frac{1}{r}\int \frac{du}{u} = \frac{1}{r}\ln|u| + C = \frac{1}{r}\ln|rP - c| + C$$

Now we integrate each side of equation (1) (the first version). We get:

$$\frac{1}{r}\ln|rP - c| = t + C \tag{2}$$

(where r, c are given in the problem, and C is the constant of integration – to be found depending on the initial value). It follows that

$$\ln|rP - c| = rt + C \tag{3}$$

(rename the constant C - the value of the constant C in equation (3 is equal to rC with the value of C from equation (2 - but this does not matter, it is still a constant value which we will find at the end of the process, depending on the initial value P_0) Now exponentiate:

$$|rP - c| = e^{rt+C} = e^C e^{rt} \tag{4}$$

Rename $e^C = K$ so we have $|rP - c| = Ke^{rt}$.

Now $|rP - c|$ means the absolute value, so it is equal to $rP - c$ if this expression is positive, but it is $-rP + c$ if it is negative. So if we start with an initial value $P_0 > \frac{c}{r}$, then $rP - c$ will be positive, and equation (4) tells us

$$rP - c = Ke^{rt} \tag{5}$$

so when we solve for P we get

$$P = \frac{c + Ke^{rt}}{r} \tag{6}$$

In order to find the value of the constant K, plug in $t = 0$ on each side of equation (5). We find $K = rP_0 - c$.

Note that when we graph the function $P = P(t)$ found in equation (6) we see a graph with unlimited growth. This corresponds to unstable equilibrium and initial value above equilibrium.

When we start with an initial value $P_0 < \frac{c}{r}$, then $rP - c$ will be negative, and equation (4) tells us

$$c - rP = Ke^{rt} \tag{7}$$

When we solve for P we get

$$P = \frac{c - Ke^{rt}}{r} \tag{8}$$

and in order to find the value of the constant K we plug in $t = 0$ in each side of equation (7) and we find that $K = c - rP_0$.

Note that when we graph the function $P = P(t)$ found in equation (8) we see a graph that decays to extinction. This corresponds to unstable equilibrium and initial value below equilibrium.

Now consider the other version of the affine equation, $\frac{dP}{dt} = c - rP$.

Separate the variables:

$$\frac{dP}{c - rP} = dt$$

Then integrate:

$$\int \frac{dP}{c - rP} = \int dt$$

Use u-substitution with $u = c - rP$ so $du = -r\,dP$. This means that we replace the symbol dP in the integral on the left hand side by $-\frac{1}{r}\,du$. So

$$\int \frac{dP}{c - rP} = -\frac{1}{r}\int \frac{du}{u} = -\frac{1}{r}\ln|u| + C = -\frac{1}{r}\ln|c - rP| + C$$

The rest of the calculation follows along the same steps as in the first case.