Math 172 Fall 2012 Handout 3 August 30

Affine equations

Affine equations can have the form $\Delta P = rP - c$ or $\Delta P = -rP + c$ ($\frac{dP}{dt}$ may be used instead of ΔP if we study a continuous process rather than a discrete one; but most of today's discussion does not depend on whether the process is discrete or continuous).

The situation that is described by these equations is a population that has a certain intrinsic per capita growth or decay rate (r), combined with a constant rate of immigration or emigration (c individuals per unit of time). Other situations that can be modeled by this type of equations include the amount of drug in the bloodstream of a patient, if the drug is administered at a constant rate (c) but a certain percentage of the drug (r) is eliminated from the bloodstream through the kidneys.

Equilibrium: (also called **stable state**) An equilibrium value for a population is a value P_{equil} that is maintained with no change in the size of the population (if that value is reached). Specifically, if $P = P_{equil}$, then $\Delta P = 0$ (or $\frac{dP}{dt} = 0$). To find the equilibrium value(s), set the expression for ΔP or $\frac{dP}{dt}$ equal to zero and solve for P).

For the exponential model: $\Delta P = rP$ for equilibrium we set $rP = 0$ so $P_{equil} = 0$. The only way in which a population with exponential growth/decay is stable is if it is equal to zero.

For an affine model $\Delta P = rP - c$ or $\Delta P = -rP + c$ set the right hand side of the equation equal to zero to find $P_{equil} = \frac{c}{r}$. This is the size of the population when the net number of individuals born into the population is exactly equal to the number of emigrants (for equations of the form $\Delta P = rP - c$), or the net number of deaths is exactly equal to the number of immigrants coming in (for equations of the form $\Delta P = -rP + c$).

Stable and unstable equilibrium; long term outcome: The long term outcome of the population depends on whether the equilibrium is stable or unstable; in case of unstable equilibrium, the long term outcome also depends on whether the initial value is larger than equilibrium or smaller than equilibrium.

Stable equilibrium means that a small change in the size of the population from its equilibrium value will cause the population to revert back toward equlibrium. Specifically, if $P > P_{equil}$ then $\Delta P < 0$ and if $P < P_{equil}$ then $\Delta P > 0$. Populations exhibiting intrinsic decline combined with immigration ($\Delta P = -rP + c$) have stable equilibrium. We can establish this based on analyzing the sign of ΔP when P is chosen above or below the equilibrium value of $\frac{c}{r}$. In the long run, populations that are modeled by this type of equations will approach the equilibrium value, no matter what initial value they start at.

Unstable equilibrium means that a small change in the size of the population from equilibrium will cause the population to move farther away from equilibrium. In other words, if $P > P_{equil}$ then $\Delta P > 0$ and if $P < P_{equil}$ then $\Delta P < 0$. Populations exhibiting intrinsic growth combined with emigration ($\Delta P = rP - c$) have unstable equilibrium. We can establish this based on analyzing the sign of ΔP when P is chosen above or below the equilibrium value of $\frac{c}{r}$. In the long run, populations that are modeled by this type of equations will become extinct if $P(0) < P_{equil}$ and will grow with no bound if $P(0) > P_{equil}$

Using the calculator to find numerical values for the discrete affine model: First change your calculator to sequence mode (SEQ) from function mode (FUNC) by using the MODE button and selecting SEQ. Push the $y =$ button in order to enter the information about your sequence (discrete model). The calculator uses n instead of t and u, v, etc. instead of P. Make $nMin = 0$ on the first line. The next line you need to enter is $u(n)$. This should be a recursive equation, showing how to get from $u(n-1)$ to $u(n)$.

Example: Say the equation is $\Delta P = 0.5P - 20$. This means that

$$P(n+1) - P(n) = 0.5P(n) - 20$$

In order to have a recursive equation, move the $P(n)$ on the left hand side to the right. Now we have

$$P(n+1) = 1.5P(n) - 20$$

This is a recursive equation, but since the calculator requires that we enter $u(n)$ instead of $u(n+1)$ we need to replace $n+1$ by n and n by $n-1$. So the equation we enter in the calculator will be

$$u(n) = 1.5u(n-1) - 20$$

(**note:** you can find n on the button labeled X, T, Θ, n - the calculator will know to use n when you are in sequence mode); you can find u on top of the button labeled 7.

On the next line you need to enter the initial value ($u(nMin)$ represents the value $P(0)$). Let's say that the initial value is $P(0) = 50$. Now you have finished entering the necessary information. Hit enter and leave the rest blank (the rest of the lines will be used if we want to enter two different sequences - we will do this later). Now hit 2ND and GRAPH in order to produce a table of values. You will see that the values of u increase rapidly (indeed the equation indicates unstable equilibrium and the initial value we chose is larger than equilibrium, so the increase was predicted by the theoretical discussion). You might be asked (for example) to predict when this population will reach 1000 individuals or more. Just scroll down the table until you see a value of $u > 1000$ and read off the corresponding n ($n = 12$).

Now change the initial value to be $P(0) = 30$. You do this by hitting $y =$ and changing the $u(nMin)$ value. Now look at the table. You will see that the population becomes extinct at $t = 4$ (reflected in negative values in the table - the model is no longer valid once the population has become extinct, since there are no more individuals left to emigrate). Again, this was predicted by the theoretical discussion.