Math 172 Fall 2012 Handout 2 August 28

Soving the continuous exponential model equation — the method of separation of variables:

The continuous exponential model equation is:

$$\frac{dP}{dt} = rP$$

where r is the per capita growth rate (it is a specified constant).

We think of t and P as the *variables* in the equation. Whenever the right hand side of a differential equation is a *product* $f(t)g(P)$, so that the equation has the form

$$\frac{dP}{dt} = f(t)g(P)$$

we can separate the variables as follows:

$$\frac{dP}{g(P)} = f(t)\,dt$$

and then we can integrate:

$$\int \frac{g(P)}{dP} = \int f(t)\,dt$$

In case of the continous model equation, separation of variables leads to

$$\frac{dP}{P} = r\,dt$$

Integrating leads to:

$$\ln(P) = rt + C$$

where C is a constant (to be determined depending on the initial value). Threfore

$$P = e^{\ln(P)} = e^{rt+C} = e^C e^{rt}.$$

When $t = 0$, $P(0) = e^C$, so the general solution is $P(t) = P(0)e^{rt}$.

Doubling time The doubling time is one of the features of the exponential model. It means the length of time needed for the population to double in size.In an exponential model, this time is always the same (depending only on the value r of the per capita growth rate) regardless of the initial value of the population.

To find the doubling time:

- **graphical method:** graph the population function $P = P_0(1+r)^t$ (for the discrete model) or $P = P_0e^{rt}$ on your graphing calculator; use y instead of P and x instead of t, and let P_0=1. Trace along your graph until you reach $y = 2$ and read the corresponding x-coordinate of that point.

- **algebraic method:** set $P = 2P_0$ and solve for t by taking the ln of each side.
  the discrete model: $P_0(1+r)^t = 2P_0$; $(1+r)^t = 2$
  $t\ln(1+r) = \ln 2$, so $t = \dfrac{\ln 2}{\ln(1+r)}$.

  the continous model: $P_0e^{rt} = 2P_0$; $e^{rt} = 2$
  $rt = \ln 2$, so $t = \dfrac{\ln 2}{r}$.

Discrete vs. continous exponential model: which grows faster?

Consider an exponential model with a per capita growth rate of 2%, or $r = 0.02$.

If the discrete model is used, we have $P = P_0 1.02^t$.

If the continous model is used, we have $P = P_0e^{.02t}$.

Let $P_0 = 100$. In order to see which of the two models grows faster, let's see the values of $P(20)$: $P(20) = 148.6$ for the discrete model; $P(20) = 149.2$ for the continous model. So the continous model achieves a larger value in the same amount of time. Let's also compare the doubling times: $t_{dbl} = 35$ for the discrete model; $t_{dbl} = 34.6$ for the continous model. Thus the continous model doubles faster and we conclude that given the same per capita growth rate r, the continous model grows faster than the discrete model.

Affine models

A discrete affine model is given by a difference equation

$$\Delta P = rP + c$$

where r and c are constants.

A continous affine model is given by a differential equation

$$\frac{dP}{dt} = rP + c$$

where r and c are constant.

The numerical values of r and c could be positive or negative. Affine models occur for instance in a case of a population that has an intrinsic per capita growth rate r (or decay rate, if r is negative), and in addition to the intrinsic growth or decay there is a constant amount of immigration (if c is positive) or emigration (if c is negative). Problems 1 and 3 on Worksheet 2 are also examples of affine models.

Numerical behavior for discrete affine models: We will study discrete affine models from a numerical point of view. We will use the calculator in order to produce values of $P(t)$ for various values of t, and draw conclusions about the long term behavior of P.

Please click on the link for the instructions on how to enter sequences on the calculator at http://www.math.uiuc.edu/ murphyrf/teaching/M172-S2009/instructions.html

Note that in order to enter a *sequence* $P(t)$ in the calculator, we need to write it in the form of a recursive equation, so

$P(t+1) = P(t)+\Delta P$ this is the general form of a recursive equation in our case:

$$P(t+1) = P(t) + rP(t) + c$$

Note that when working with sequences on the calculator we need to use n in the place of t and u or v as the names for the sequences. Moreover, we will give the value of $P(n)$ in terms of the value of $P(n-1)$, so we let $t+1 = n$, $t = n-1$ in the above equation. We will write the equation in the form:

$$u(n) = u(n-1) + ru(n-1) + c$$

or

$$u(n) = (1+r)u(n-1) + c$$