Math 172 Fall 2012 Handout 15

Predator-Prey Models: the functional response of the predators

The general form of the predator-prey equations can be written as

$$\frac{dV}{dt} = \text{intrinsic growth} - R(V)P$$

$$\frac{dP}{dt} = -qP + \beta PV$$

The intrinsic growth component in the first equation can be exponential (rV) or logistic ($rV(1 - \frac{V}{K})$).

$R(V)$ is the functional response of the predators (also called feeding rate) and it means the amount of victims consumed per predator, as a function of the number of victims that are present. A functional response of type I is $R(V) = \alpha V$ (which was used in the basic model); this does not take into consideration that saturation occurs when a large number of victims are present. In order to take the saturation effect into consideration, we will use a functional response of type II: $R(V) = \frac{kV}{V+D}$ or a functional response of type III:$R(V) = \frac{kV^2}{V^2+D^2}$. In the type II or type III formula, k represents the saturation constant (the maximum number of victims that can be consumed per predator) and D represents the half-saturation constant (the number of victims that need to present in order for the feeding rate to reach half of its maximum. When $R(V)$ given by a type II or type III formula is graphed as function of V, you should see a portion that is almost linear for small values of V, followed by a portion where $R(V)$ grows slower and slower and eventually stabilizes at k when V is large ($y = k$ is a horizontal asymptote for the graph).

The equilibrium pairs for the general predator-prey systems are: $(0,0)$, $(K,0)$ (if the growth component in the dV/dt equation is logistic), and possibly an equilibrium value with both populations present. In order to find this latter equilibrium value, we need to set both equation equal to zero. The second equation always gives $V = q/\beta$ (when $P \neq 0$). We then plug this value of V into the first equation and solve for P.

For example consider the situation when the first equation is

$$\frac{dV}{dt} = rV - cV^2 - \frac{kV}{D+V}P$$

(logistic intrinsic growth for the victims, functional response of type II) We set this equal to zero and factor out V:

$$V(r - cV - \frac{k}{D+V}P) = 0$$

Since $V \neq 0$ (we are looking for the equilibrium with both populations present) it follows that the paranthesis must be zero, so

$$r - cV - \frac{k}{D+V}P = 0,$$

in other words $P = \dfrac{(r-cV)(D+V)}{k}$ and the numerical value of P is obtained once we plug in $V = q/\beta$.

The isocline for P is always the line of equation $V = q/\beta$ (a vertical line). The isocline for V depends on the type of equation being considered, and sometimes might not be a line. For example with the equation for dV/dt considered above, the iscoline for V has equation

$$r - cV - \frac{k}{D+V}P = 0$$

which we can write as $P = \frac{(r-cV)(D+V)}{k}$. This is an equation for an upside-down parabola, therefore in this case the isocline for V is a portion of a parabola. The meaning of this isocline is as follows: in order for the V populaiton to be maintained at a constant level, the point (V, P) must be on the parabola. Note that the value for P needed for this to occur gets smaller when V is large; this is due to the intrinsic logistic behavior of V, which means that the growth of V is restricted by the presence of a large number of V's.

If we consider an equation in which V has intrinsic exponential growth, and the predators have a type II functional response, the equation becomes

$$\frac{dV}{dt} = rV - \frac{kV}{V+D}P$$

and the isocline for V is the equation obtained by setting this equal to zero and factoring out V:

$$V(r - \frac{k}{V+D}P) = 0$$

therefore the equation of the isocline is the line of equation

$$r(V+D) - kP = 0$$

Note that in this model more and more predators are needed to keep the V population at a constant level when V gets larger.