Math 172 Fall 2012 Handout 13 September 18

Predator-Prey Models

We will consider two populations: the predators (P) and the victims (V).

Basic Assumptions:

- In the absence of the victims, the predator population decays exponentially (due to lack of food)
- In the absence of the predators, the victim population will either increase exponentially, or follow the logistic model.

Thus, the equations are

$$\frac{dP}{dt} = -qP + \text{contribution due to the presence of victims}$$

$$\frac{dV}{dt} = rP - \text{effect of predators}$$

or

$$\frac{dV}{dt} = rV\left(1 - \frac{V}{K}\right) - \text{effect of predators}$$

Several variations of the model will be considered.

In the most basic version, we assume that every time when a P gets sufficiently close to a V, the V is eaten. Moreover, the probability for a P to get sufficiently close to a V is proportional to VP, and the effect that the consumption of the V's by the P's contributes to the ability of the P's to reproduce in an amount proportional to the amount of V's that are consumed. In other words, the equations for the most basic model are:

$$\frac{dV}{dt} = rV - \alpha VP$$

$$\frac{dP}{dt} = -qP + \beta VP$$

where r, q, α, β are constants (usually specified in the problems).

To find the equilibrium, we must set both $\frac{dV}{dt}$ and $\frac{dP}{dt}$ equal to zero:

$$V(r - \alpha P) = 0$$

$$P(-q + \beta V) = 0$$

The values that make both equations true are: $(V = 0, P = 0)$ and $(V = q/\beta, P = r/\alpha)$.

For the state space, we will represent values of V on the horizontal axis, and values of P on the vertical axis. The isocline for V is the line of equation $P = r/\alpha$ (this is the equation that makes $dV/dt = 0$), which will be represented as a horizontal line in the state space. The isocline for P is the line of equation $V = q/\beta$ (this is the equation that makes $dP/dt = 0$), which will be represented as a vertical line in the state space. Thus the state space consists of a horizontal line and a vertical line, which intersect at the equilibrium point $(V = q/\beta, P = r\alpha)$.

It is interesting to note that the equation for the isocline of V depends only on P. This means that a fixed number of predators ($P = r/\alpha$) is needed in order to keep the victim population in check at a constant level. Similarly, the equation for the isocline of P depends only on V. This means that a fixed number of victims ($V = q/\beta$) in order to allow for the predators to sustain themselves at a constant level.

Now we analyze the behavior of the two populations in terms of the position in the state space. Note that the state space is divided by the isoclines into four regions.

In region I we have $V > q/\beta, P > r/\alpha$ which causes V to decrease and P to increase. In region II we have $V < q/\beta, P > r/\alpha$ which causes both V and P to decrease. In region III we have $V < \beta/q$, $P < r/\alpha$, which causes V to increase and P to decrease. In region IV we have $V > \beta/q, P < r/\alpha$, which causes both V and P to increase. When this behavior is represented by arrows, we see that it results in a circular behavior: the point that represents the values $(V(t), P(t)$ in the state space moves in a circle around the equilibrium point.

This leaves open the possibilities that the values might spiral inward toward equilibrium, or outward away from equilibrium, but we do not have the theoretical machinery necessary to detect this. Therefore we will have to rely on the Euler method to see what happens in each specific case based on numerical values.