Math 172 Spring 2011 Handout 10

Models in which two populations interact

We will consider two possible interactions between two populations or species: competition and predation.

In both of these models we will use two functions, $N_1(t)$ and $N_2(t)$ for the two populations, and the model will be given by two differential (or difference) equations, one describing dN_1/dt and one describing dN_2/dt. Each of dN_1/dt and dN_2/dt will be described in terms of both N_1 and N_2. In other words, the rate of change of each population is influenced by the sizes of both populations.

The competition model; Lotka-Volterra equations

We assume that the two populations N_1, N_2 are in competition for the same resources. This means that the growth of each population is inhibited by the presence of the other. Since the resources are assumed to be limited, each population would follow a **logistic** model **in the absence of the other**:

$$\frac{dN_1}{dt} = r_1 N_1 \frac{K_1 - N_1}{K_1}$$
$$\frac{dN_2}{dt} = r_2 N_2 \frac{K_2 - N_2}{K_2}$$

Population 1 would have carrying capacity K_1, and population 2 would have carrying capacity K_2.

The negative effect that the presence of each population has on the growth of the other translates into a decrease in the per capita growth rate of each population in an amount proportional to the size of the other:

$$\frac{dN_1}{dt} = r_1 N_1 \frac{K_1 - N_1 - \alpha N_2}{K_1}$$
$$\frac{dN_2}{dt} = r_2 N_2 \frac{K_2 - N_2 - \beta N_1}{K_2}$$

These are the Lotka-Volterra equations for the competition model.

The constants α, β are called the **competition coefficients**. α is a measure of the (negative) effect that species 2 has on species 1.Similarly, β is a measure of the (negative) effect that species 1 has on species 2.

The values of the competition coefficients α, β as well as the carrying capacities K_1, K_2 of the two population determine the outcome of the competition: the two populations might coexist in stable or unstable equilibriu, or one of the populations might drive the other to extinction.

Equilibrium for competition models

In the context of two population that interact, equilibrium means a pair of values (N_1, N_2) that causes each population to not change. In order to find the equilibrium for the Lotka-Volterra competition model, we set each of the right hand sides equal to zero; we have

$$\begin{cases} N_1(K_1 - N_1 - \alpha N_2) & = & 0 \\ N_2(K_2 - N_2 - \beta N_1) & = & 0 \end{cases}$$

The first equation implies either $N_1 = 0$ or $N_1 + N_2 = K_1$. The second equation implies either $N_2 = 0$ or $N_2 + \beta N_1 = 0$. Thus $N_1 = 0, N_2 = 0$ is one of the equilibrium values (where neither one of the populations is present). The other possiblities are: $(N_1 = 0, N_2 = K_2)$, $(N_1 = K_1, N_2 = 0)$, or, if an equilibrium in which both populations are present is possible:

$$\begin{cases} N_1 + \alpha N_2 & = & K_1 \\ N_2 + \beta N_1 & = & K_2 \end{cases}$$

The solution is

$$N_1^* = \frac{K_2 - \beta K_1}{1 - \alpha\beta} \quad N_2^* = \frac{K_1 - \alpha K_2}{1 - \alpha\beta}$$

Note that the equilibrium values one gets for the competition model are smaller for each population than the equilibrium that population would reach in the absence of the other: $N_1^* < K_1, N_2^* < K_2$.

Also note that it is possible for the values of N_1^* or N_2^* (or both) to be negative. If this happens, it means that the two populations cannot coexist in equilibrium.