Eigenvalues and Eigenvectors of a matrix

Let A be an $n \times n$ matrix and B a $n \times 1$ vector.

We say that the vector B is an *eigenvector* for the matrix A if there exists a constant λ (called the *eigenvalue* associated to the eigenvector B) such that

$$A \cdot B = \lambda B$$

In other words, the requirement for B to be an eigenvector for the matrix A is that the vector $A \cdot B$ is **proportional** to B, and the corresponding eigenvalue λ is the constant of proportionality.

Examples: Consider

$$A = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ -5 \end{bmatrix} \quad C = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

We have

$$A \cdot B = \begin{bmatrix} -15 \\ 25 \end{bmatrix}$$

This vector is not proportional to B since $-15/5 \neq 25/-5$. Thus B is not an eigenvector for the matrix A.

$$A \cdot C = \begin{bmatrix} 21 \\ 21 \end{bmatrix}$$

This vector is proportional to C: $21/3 = 21/3 = 7$, in other words $A \cdot C = 7C$. The corresponding eigenvalue is $\lambda = 7$.

Now we return to the study of populations with age structures.

Recall that the population at time n is represented by a vector

$$B_n = \begin{bmatrix} C_n \\ M_n \\ O_n \end{bmatrix}$$

which satisfies the recursive equation $B_{n+1} = A \cdot B_n$, where A is the transition matrix.

We will explore how the eigenvalues and eigenvectors of the transition matrix A relate to the notions of stable state for the population vector, and exponential behavior for the total population size.

Recall the definitions:

The distribution vector at step n is

$$D_n = \begin{bmatrix} C_n/P_n \\ M_n/P_n \\ O_n/P_n \end{bmatrix}$$

where $P_n = C_n + M_n + O_n$.

We say that the population has reached a **stable state** at step n if $D_n = D_{n+1}$. That is, if the distribution vector does not change when we go to the next step. The distribution vector D_n achieved when the population reaches a stable state is called the **stable distribution** of the population.

We say that the total size of the population has exponential behavior with per-capita growth rate r if $\frac{P_{n+1}}{P_n} = 1+r$ whenever n is large enough (this means that the ratio $\dfrac{P_{n+1}}{P_n} = 1+r$ stabilizes at a constant value).

Observation: If the population has reached stable state at step n, this means that the vector B_n is an eigenvector for the matrix A.

Indeed, stable state means that the vectors B_n/P_n and B_{n+1}/P_{n+1} coincide, where by B_n/P_n we mean the vector obtained by dividing each entry in B_n by the total population size P_n. Recalling that $B_{n+1} = AB_n$, this equation can be written as

$$\frac{AB_n}{P_{n+1}} = \frac{B_n}{P_n},$$

or equivalently

$$AB_n = \frac{P_{n+1}}{P_n} \cdot B_n$$

which means that B_n is an eigenvector for A with corresponding eigenvalue equal to the ratio P_{n+1}/P_n.

Typically, the transition matrix A will have three eigenvalues $\alpha_1, \alpha_2, \alpha_3$, and only one of these eigenvalues will be larger than 1. Say that $\alpha_1 > 1$ and $< -1\alpha_2, \alpha_3 < 1$. We will refer to α_1 as the **dominant eigenvalue**. Let's say that the eigenvectors corresponding to $\alpha_1, \alpha_2, \alpha_3$ are U_1, U_2, U_3 respectively so that we have $A \cdot U_1 = \alpha_1 U_1, A \cdot U_2 = \alpha_2 V_2, A \cdot U_3 = \alpha_3 U_3$. Further, the initial population vector B_0 can usually be expressed as a linear combination of the eigenvectors:

$$B_0 = c_1 U_1 + c_2 U_2 + c_3 U_3$$

It follows that

$$\begin{aligned} B_1 = A \cdot B_0 = A(c_1 U_1 + c_2 U_2 + c_3 U_3) &= c_1(AU_1) + c_2(AU_2) + c_3(AU_3) \\ &= (c_1\alpha_1)U_1 + (c_2\alpha_2)U_2 + (c_3\alpha_3)U_3 \end{aligned}$$

$$\begin{aligned} B_2 = AB_1 = A(c_1\alpha_1 U_1 + c_2\alpha_2 U_2 + c_3\alpha_3 U_3) &= c_1\alpha_1(AU_1) + c_2\alpha_2(AU_2) + c_3\alpha_3(AU_3) \\ &= c_1\alpha_1^2 U_1 + c_2\alpha_2^2 U_2 + c_3\alpha_3^2 U_3 \end{aligned}$$

and in general

$$B_n = c_1\alpha_1^n U_1 + c_2\alpha_2^n U_2 + c_3\alpha_3^n U_3$$

Since $-1 < \alpha_2, \alpha_3 < 1$, their powers α_2^n, α_3^n will approach zero when n is very large, therefore the entries in $c_2\alpha_2^n U_2 + c_3\alpha_3^n U_3$ become negligible in the long run, and we can approximate

$$B_n \cong c_1\alpha_1^n U_1 \quad \text{for n sufficiently large}$$

This means that B_n will be an eigenvector for the transition matrix A when n is sufficiently large, and therefore we will have exponential behavior ($P_{n+1} = \alpha_1 P_n$) for the total population size and stable distribution vectors from that point on.