1. Without solving the integral, find the appropriate change of variables and simplify the integral.

(a) $\displaystyle\int \left(36-9x^2\right)^{5/2}\,dx$

$$36-9x^2 = 36\left[1-\frac{9}{36}x^2\right] = 36\left[1-\left(\frac{3}{6}x\right)^2\right]$$

The appropriate substitution is $\frac{3}{6}x = \sin\theta$, with $dx = \frac{6}{3}\cos\theta\,d\theta$, and the integral becomes

$$\begin{aligned}
\int \left(36-9x^2\right)^{5/2}\,dx &= \int \left[36-9\left(\frac{6}{3}\sin\theta\right)^2\right]^{5/2}\left[\frac{6}{3}\cos\theta\,d\theta\right]\\
&= \int \left[36-9\left(\frac{36}{9}\sin^2\theta\right)\right]^{5/2}\left[\frac{6}{3}\cos\theta\,d\theta\right]\\
&= \int \left[36-36\sin^2\theta\right]^{5/2}\left[\frac{6}{3}\cos\theta\,d\theta\right] = \int \left[36\left(1-\sin^2\theta\right)\right]^{5/2}\left[\frac{6}{3}\cos\theta\,d\theta\right]\\
&= \int \left[36\left(\cos^2\theta\right)\right]^{5/2}\left[\frac{6}{3}\cos\theta\,d\theta\right] = \int \left[(6\cos\theta)^2\right]^{5/2}\left[\frac{6}{3}\cos\theta\,d\theta\right]\\
&= \int (6\cos\theta)^5\left[\frac{6}{3}\cos\theta\,d\theta\right]\int \left(6^5\cos^5\theta\right)\left[\frac{6}{3}\cos\theta\,d\theta\right]\\
&= \frac{6^6}{3}\int \cos^6\theta\,d\theta.
\end{aligned}$$

(b) $\displaystyle\int \frac{1}{\sqrt{x^2+20}}\,dx$

$$x^2+20 = 20\left[\frac{x^2}{20}+1\right] = 20\left[\left(\frac{x}{\sqrt{20}}\right)^2+1\right]$$

The appropriate substitution is $\frac{x}{\sqrt{20}} = \tan\theta$, with $dx = \sqrt{20}\sec^2\theta\,d\theta$, and the integral becomes

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+20}}\,dx &= \int \frac{1}{\sqrt{\left(\sqrt{20}\tan\theta\right)^2+20}}\left[\sqrt{20}\sec^2\theta\,d\theta\right]\\
&= \int \frac{1}{\sqrt{\left(20\tan^2\theta\right)+20}}\left[\sqrt{20}\sec^2\theta\,d\theta\right]\\
&= \int \frac{1}{\sqrt{20\left(\tan^2\theta+1\right)}}\left[\sqrt{20}\sec^2\theta\,d\theta\right] = \int \frac{1}{\sqrt{20\left(\sec^2\theta\right)}}\left[\sqrt{20}\sec^2\theta\,d\theta\right]\\
&= \int \frac{1}{\sqrt{20}\sec\theta}\left[\sqrt{20}\sec^2\theta\,d\theta\right]\\
&= \int \sec\theta\,d\theta.
\end{aligned}$$

(c) $\displaystyle\int \sqrt{x^2+2x+10}\,dx$

Complete the square,

$$\begin{aligned} x^2+2x+10 = \left(x^2+2x+1\right)+10-1 &= (x+1)^2+9 \\ &= 9\left[\frac{(x+1)^2}{9}+1\right] = 9\left[\left(\frac{x+1}{3}\right)^2+1\right] \end{aligned}$$

The appropriate substitution is $\dfrac{x+1}{3}=\tan\theta$, with $dx = 3\sec^2\theta\,d\theta$, and the integral becomes

$$\begin{aligned} \int \sqrt{x^2+2x+10}\,dx = \int \sqrt{(x+1)^2+9}\,dx &= \int \sqrt{(3\tan\theta)^2+9}\,\left[3\sec^2\theta\,d\theta\right] \\ &= \int \sqrt{9\tan^2\theta+9}\,\left[3\sec^2\theta\,d\theta\right] \\ &= \int \sqrt{9\left(\tan^\theta+1\right)}\,\left[3\sec^2\theta\,d\theta\right] \\ &= \int \sqrt{9\sec^2\theta}\,\left[3\sec^2\theta\,d\theta\right] \\ &= \int \left[3\sec\theta\right]\,\left[3\sec^2\theta\,d\theta\right] \\ &= \int 9\sec^3\theta\,d\theta. \end{aligned}$$

(d) $\displaystyle\int \left(9-4t^2\right)^{3/2}\,dt$

$$9-4t^2 = 9\left[1-\frac{4}{9}t^2\right] = 9\left[1-\left(\frac{2}{3}t\right)^2\right]$$

The appropriate substitution is $\dfrac{2}{3}t=\sin\theta$, with $dt=\dfrac{3}{2}\cos\theta\,d\theta$, and the integral becomes

$$\begin{aligned} \int \left(9-4t^2\right)^{3/2}\,dt &= \int \left[9-4\left(\frac{3}{2}\sin\theta\right)^2\right]^{3/2}\left[\frac{3}{2}\cos\theta\,d\theta\right] \\ &= \int \left[9-4\left(\frac{9}{4}\sin^2\theta\right)\right]^{3/2}\left[\frac{3}{2}\cos\theta\,d\theta\right] \\ &= \int \left[9-9\sin^2\theta\right]^{3/2}\left[\frac{3}{2}\cos\theta\,d\theta\right] \\ &= \int \left[9\left(1-\sin^2\theta\right)\right]^{3/2}\left[\frac{3}{2}\cos\theta\,d\theta\right] \\ &= \int \left[9\left(\cos^2\theta\right)\right]^{3/2}\left[\frac{3}{2}\cos\theta\,d\theta\right] \\ &= \int \left[(3\cos\theta)^2\right]^{3/2}\left[\frac{3}{2}\cos\theta\,d\theta\right] = \int \left[3\cos\theta\right]^3\left[\frac{3}{2}\cos\theta\,d\theta\right] \\ &= \int \frac{3^4}{2}\cos^4\theta\,d\theta \end{aligned}$$

(e) $\displaystyle\int \frac{x^2+4}{\sqrt{x^2-4}}\,dx$

$$x^2-4=4\left[\frac{x^2}{4}-1\right]=4\left[\left(\frac{x}{2}\right)^2-1\right]$$

The appropriate substitution is $\frac{x}{2}=\sec\theta$, with $dx=2\sec\theta\,\tan\theta\,d\theta$, and the integral becomes

$$\begin{aligned}\int \frac{x^2+4}{\sqrt{x^2-4}}\,dx &= \int \frac{(2\sec\theta)^2+4}{\sqrt{(2\sec\theta)^2-4}}\,[2\sec\theta\,\tan\theta\,d\theta]\\ &= \int \frac{4\sec^2\theta+4}{\sqrt{4\sec^2\theta-4}}\,[2\sec\theta\,\tan\theta\,d\theta]\\ &= \int \frac{4\sec^2\theta+4}{\sqrt{4\left(\sec^2\theta-1\right)}}\,[2\sec\theta\,\tan\theta\,d\theta]\\ &= \int \frac{4\sec^2\theta+4}{\sqrt{4\left(\tan^2\theta\right)}}\,[2\sec\theta\,\tan\theta\,d\theta]\\ &= \int \frac{4\sec^2\theta+4}{2\tan\theta}\,[2\sec\theta\,\tan\theta\,d\theta]=\int\left[4\sec^2\theta+4\right]\,[\sec\theta\;d\theta]\\ &= \int\left(4\sec^3\theta+4\sec\theta\right)\,d\theta.\end{aligned}$$

(f) $\displaystyle\int e^{4x}\sqrt{e^{8x}-9}\,dx$ Use the u-substitution $u=e^{4x}$, which gives $du=4e^{4x}\,dx$. With these the integral becomes

$$\int e^{4x}\sqrt{e^{8x}-9}\,dx=\frac{1}{4}\int\sqrt{u^2-9}\,du$$

$$u^2-9=9\left[\frac{u^2}{9}-1\right]=9\left[\left(\frac{u}{3}\right)^2-1\right]$$

The appropriate substitution is $\frac{u}{3}=\sec\theta$, with $du=3\sec\theta\,\tan\theta\,d\theta$, and the integral becomes

$$\begin{aligned}\int e^{4x}\sqrt{e^{8x}-9}\,dx=\frac{1}{4}\int\sqrt{u^2-9}\,du &= \frac{1}{4}\int\sqrt{(3\sec\theta)^2-9}\,[3\sec\theta\,\tan\theta\,d\theta]\\ &= \frac{1}{4}\int\sqrt{9\sec^2\theta-9}\,[3\sec\theta\,\tan\theta\,d\theta]\\ &= \frac{1}{4}\int\sqrt{9\left(\sec^2\theta-1\right)}\,[3\sec\theta\,\tan\theta\,d\theta]\\ &= \frac{1}{4}\int\sqrt{9\tan^2\theta}\,[3\sec\theta\,\tan\theta\,d\theta]\\ &= \frac{1}{4}\int[3\tan\theta]\,[3\sec\theta\,\tan\theta\,d\theta]\\ &= \frac{9}{4}\int\sec\theta\tan^2\theta\;d\theta.\end{aligned}$$

(g) $\displaystyle\int \frac{z}{(1-4z-2z^2)^{3/2}}\,dz$

First, complete the square

$$\begin{aligned} 1-4z-2z^2 = 1-2\left(z^2+2z+1\right)+2 &= 3-2(z+1)^2 \\ &= 3\left[1-\frac{2}{3}(z+1)^2\right] = 3\left[1-\left(\frac{\sqrt{2}}{\sqrt{3}}(z+1)\right)^2\right] \end{aligned}$$

The appropriate substitution is $\frac{\sqrt{2}}{\sqrt{3}}(z+1)=\sin\theta$, with $dz=\frac{\sqrt{3}}{\sqrt{2}}\cos\theta\,d\theta$, and the integral becomes

$$\begin{aligned} \int \frac{z}{(1-4z-2z^2)^{3/2}}\,dz &= \int \frac{z}{\left(3\left[1-\left(\frac{\sqrt{2}}{\sqrt{3}}(z+1)\right)^2\right]\right)^{3/2}}\,dz \\ &= \int \frac{\frac{\sqrt{3}}{\sqrt{2}}\sin\theta-1}{\left(3\left[1-\sin^2\theta\right]\right)^{3/2}}\left[\frac{\sqrt{3}}{\sqrt{2}}\cos\theta\,d\theta\right] \\ &= \frac{3}{2}\int \frac{\sin\theta-1}{(3\cos^2\theta)^{3/2}}\cos\theta\,d\theta \\ &= \frac{3}{2}\cdot\frac{1}{3^{3/2}}\int \frac{\sin\theta-1}{\cos^3\theta}\cos\theta\,d\theta \\ &= \frac{1}{2\sqrt{3}}\int \frac{\sin\theta-1}{\cos^2\theta}\,d\theta. \end{aligned}$$

(h) $\displaystyle\int \frac{1}{\sqrt{9x^2-36x+37}}\,dx$

First, complete the square

$$9x^2-36x+37 = 9\left(x^2-4x+4\right)+37-36 = 9(x-2)^2+1 = (3(x-2))^2+1$$

The appropriate substitution is $3(x-2)=\tan\theta$, with $dx=\frac{1}{3}\sec^2\theta\,d\theta$, and the integral becomes

$$\begin{aligned} \int \frac{1}{\sqrt{9x^2-36x+37}}\,dx = \int \frac{1}{\sqrt{(3(x-2))^2+1}}\,dx &= \int \frac{1}{\sqrt{\tan^2\theta+1}}\left[\frac{1}{3}\sec^2\theta\,d\theta\right] \\ &= \frac{1}{3}\int \frac{1}{\sqrt{\sec^2\theta}}\sec^2\theta\,d\theta \\ &= \frac{1}{3}\int \frac{1}{\sec\theta}\sec^2\theta\,d\theta \\ &= \frac{1}{3}\int \sec\theta\,d\theta. \end{aligned}$$

(i) $\int \sqrt{7w^2 - 1}\, dw$

$$7w^2 - 1 = \left(\sqrt{7}w\right)^2 - 1$$

The appropriate substitution is $\sqrt{7}w = \sec\theta$, with $dw = \frac{1}{\sqrt{7}}\sec\theta\,\tan\theta\,d\theta$, and the integral becomes

$$\begin{aligned}\int \sqrt{7w^2 - 1}\,dw &= \int \sqrt{\sec^2\theta - 1}\left[\frac{1}{\sqrt{7}}\sec\theta\,\tan\theta\,d\theta\right] = \int \sqrt{\tan^2\theta}\left[\frac{1}{\sqrt{7}}\sec\theta\,\tan\theta\,d\theta\right] \\ &= \int \tan\theta\left[\frac{1}{\sqrt{7}}\sec\theta\,\tan\theta\,d\theta\right] = \frac{1}{\sqrt{7}}\int \sec\theta\tan^2\theta\,d\theta.\end{aligned}$$

(j) $\int \cos(x)\sqrt{9 + 25\sin^2(x)}\,dx$

First do u-substitution with $u = \sin x$, which gives $du = \cos x\,dx$ and the integral becomes

$$\int \cos(x)\sqrt{9 + 25\sin^2(x)}\,dx = \int \sqrt{9 + 25u^2}\,du$$

$$9 + 25u^2 = 9\left[1 + \frac{25}{9}u^2\right] = 9\left[1 + \left(\frac{5}{3}u\right)^2\right]$$

The appropriate substitution is $\frac{5}{3}u = \tan\theta$, with $du = \frac{3}{5}\sec^2\theta\,d\theta$, and the integral becomes

$$\begin{aligned}\int \cos(x)\sqrt{9 + 25\sin^2(x)}\,dx = \int \sqrt{9 + 25u^2}\,du &= \int \sqrt{9 + 9\tan^2\theta}\left[\frac{3}{5}\sec^2\theta\,d\theta\right] \\ &= \frac{3}{5}\int \sqrt{9\sec^2\theta}\sec^2\theta\,d\theta = \frac{3}{5}\int \sec\theta\cdot\sec^2\theta\,d\theta \\ &= \frac{3}{5}\int \sec^3\theta\,d\theta\end{aligned}$$

(k) $\int \frac{1}{\sqrt{10x - x^2}}\,dx$

First, complete the square

$$10x - x^2 = -\left(x^2 - 10x + 25\right) + 25 = 25 - (x-5)^2 = 25\left[1 - \frac{(x-5)^2}{25}\right] = 25\left[1 - \left(\frac{x-5}{5}\right)^2\right]$$

The appropriate substitution is $\frac{x-5}{5} = \sin\theta$, with $dx = 5\cos\theta\,d\theta$, and the integral becomes

$$\begin{aligned}\int \frac{1}{\sqrt{10x - x^2}}\,dx = \int \frac{1}{\sqrt{25\left[1 - \left(\frac{x-5}{5}\right)^2\right]}}\,dx &= \int \frac{1}{\sqrt{25\left[1 - \sin^2\theta\right]}}\left[5\cos\theta\,d\theta\right] \\ &= \int \frac{1}{\sqrt{25\cos^2\theta}}\left[5\cos\theta\,d\theta\right] \\ &= \int \frac{1}{5\cos\theta}\left[5\cos\theta\,d\theta\right] = \int d\theta.\end{aligned}$$

2. Evaluate the following integrals

(a) $\displaystyle\int \frac{4}{\sqrt{1-x^2}}\,dx$

Use $x = \sin\theta, \qquad dx = \cos\theta\,d\theta, \qquad 1-\sin^2\theta = \cos^2\theta$

$$\begin{aligned}\int \frac{4}{\sqrt{1-x^2}}\,dx = 4\int \frac{1}{\sqrt{1-\sin^2\theta}}\cdot(\cos\theta\,d\theta) = 4\int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta &= 4\int \frac{\cos\theta}{\cos\theta}\,d\theta \\ &= 4\int d\theta \\ &= 4\theta + C = \boxed{4\sin^{-1}x + C.}\end{aligned}$$

(b) $\displaystyle\int \frac{1}{1+x^2}\,dx$

Use $x = \tan\theta, \qquad dx = \sec^2\theta\,d\theta, \qquad \tan^2\theta + 1 = \sec^2\theta$

$$\begin{aligned}\int \frac{1}{1+x^2}\,dx = \int \frac{1}{1+\tan^2\theta}\cdot\left(\sec^2\theta\,d\theta\right) = \int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta &= \int d\theta \\ &= \theta + C = \boxed{\tan^{-1}x + C.}\end{aligned}$$

(c) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$

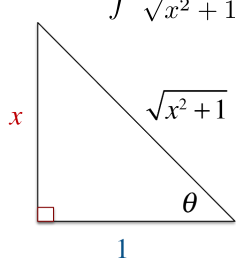
Use $x = 3\sin\theta, \qquad dx = 3\cos\theta\,d\theta, \qquad 1-\sin^2\theta = \cos^2\theta$

$$\begin{aligned}\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{1}{\sqrt{9-9\sin^2\theta}}\cdot(3\cos\theta\,d\theta) &= \int \frac{1}{3\sqrt{1-\sin^2\theta}}\cdot(3\cos\theta\,d\theta) \\ &= \int \frac{1}{3\cos\theta}\cdot(3\cos\theta)\,d\theta = \int d\theta \\ &= \theta + C = \boxed{\sin^{-1}\left(\frac{x}{3}\right) + C.}\end{aligned}$$

(d) $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx$

Use $x = \tan\theta, \qquad dx = \sec^2\theta\,d\theta, \qquad \tan^2\theta + 1 = \sec^2\theta$

$$\begin{aligned}\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{\tan\theta}{\sec\theta}\left(\sec^2\theta\,d\theta\right) &= \int \tan\theta\sec\theta\,d\theta \\ &= \sec\theta + C \\ &= \boxed{\sqrt{x^2+1} + C.}\end{aligned}$$



(e) $\displaystyle\int \frac{x^3}{\sqrt{16-x^2}}\,dx$

Use $\quad x = 4\sin\theta, \qquad dx = 4\cos\theta\,d\theta, \qquad 1-\sin^2\theta = \cos^2\theta$

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx = \int \frac{64\,\sin^3\theta}{\sqrt{16(1-\sin^2\theta)}}\cdot(4\cos\theta\,d\theta) \quad = \quad 64\int \sin^2\theta\,(\sin\,d\theta)$$

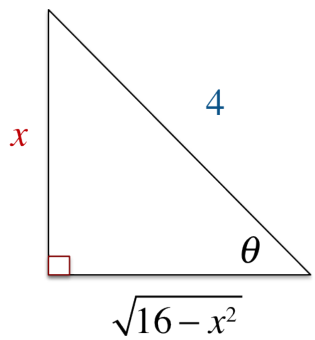
$$= 64\int\left(1-\cos^2\theta\right)(\sin\,d\theta)$$

$$= -64\int(1-u^2)\,du = -64\left(u-\frac{u^3}{3}\right)+C$$

$$= -64\cos\theta + \frac{64}{3}\cos^3\theta + C = -64\cos\theta + \frac{64}{3}(\cos\theta)^3 + C$$

$$= -64\left(\frac{\sqrt{16-x^2}}{4}\right) + \frac{64}{3}\left(\frac{\sqrt{16-x^2}}{4}\right)^3 + C$$

$$= \boxed{-16\sqrt{16-x^2} + \frac{1}{3}\left(16-x^2\right)^{3/2} + C.}$$



(f) $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$

Use $\quad x = 5\sin\theta, \qquad dx = 5\cos\theta\,d\theta, \qquad 1-\sin^2\theta = \cos^2\theta$

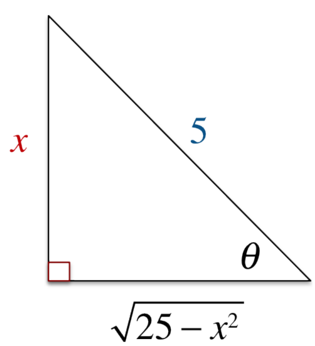
$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx = \int \frac{1}{25\sin^2\theta}\,\frac{1}{\sqrt{25(1-\sin^2\theta)}}\cdot(5\cos\theta\,d\theta) \quad = \quad \frac{1}{25}\int\frac{1}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{25}\int\csc^2\theta\,d\theta$$

$$= \frac{1}{25}(-\cot\theta) + C = -\frac{1}{25}\cot\theta + C$$

$$= -\frac{1}{25}\left(\frac{\sqrt{25-x^2}}{x}\right) + C$$

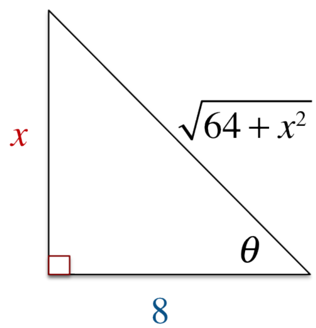
$$= \boxed{-\frac{\sqrt{25-x^2}}{25\,x} + C.}$$



(g) $\int \frac{x^3}{\sqrt{64+x^2}}\,dx$

Use $x = 8\tan\theta, \quad dx = 8\sec^2\theta\,d\theta, \quad 1+\tan^2\theta = \sec^2\theta$

$$\int \frac{x^3}{\sqrt{64+x^2}}\,dx = \int \frac{512\tan^3\theta}{\sqrt{64(1+\tan^2\theta)}}\cdot\left(8\sec^2\theta\,d\theta\right) = 512\int \frac{\tan^3\theta}{\sec\theta}\sec^2\theta\,d\theta$$



$$\begin{aligned}
&= 512\int \tan^3\theta\,\sec\theta\,d\theta\\
&= 512\int \tan^2\theta\,\left(\tan\theta\sec\theta\,d\theta\right)\\
&= 512\int \left(\sec^2\theta-1\right)\left(\tan\theta\sec\theta\,d\theta\right)\\
&= 512\int \left(u^2-1\right)du\\
&= 512\left(\frac{u^3}{3}-u\right)+C\\
&= 512\left(\frac{\sec^3\theta}{3}-\sec\theta\right)+C\\
&= 512\left(\frac{(64+x^2)^{3/2}}{3\cdot 512}-\frac{\sqrt{64+x^2}}{8}\right)+C\\
&= \boxed{\frac{(64+x^2)^{3/2}}{3}-64\sqrt{64+x^2}+C.}
\end{aligned}$$

(h) $\int \sqrt{x^2+1}\,dx$

Use $x = \tan\theta, \quad dx = \sec^2\theta\,d\theta, \quad \tan^2\theta+1 = \sec^2\theta$

$$\int \sqrt{x^2+1}\,dx = \int \sqrt{\tan^2\theta+1}\,\left(\sec^2\theta\,d\theta\right) = \int \sec^3\theta\,d\theta$$

To solve this we integrate by parts with $u = \sec\theta, \quad du = \sec\theta\tan\theta\,d\theta, \quad dv = \sec^2\theta\,d\theta \quad v = \tan\theta$

$$\begin{aligned}
\int \sec^3\theta\,d\theta &= \sec\theta\cdot\tan\theta - \int \tan\theta\cdot\left(\sec\theta\tan\theta\,d\theta\right)\\
&= \sec\theta\,\tan\theta - \int \sec\theta\tan^2\theta\,d\theta = \sec\theta\,\tan\theta - \int \sec\theta\left(\sec^2\theta-1\right)\,d\theta\\
&= \sec\theta\,\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta\\
2\int \sec^3\theta\,d\theta &= \sec\theta\,\tan\theta + \int \sec\theta\,d\theta = \sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta| + C\\
\int \sec^3\theta\,d\theta &= \frac{1}{2}\left(\sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta|\right)+C\\
\int \sqrt{x^2+1}\,dx &= \boxed{\frac{1}{2}\left(x\sqrt{x^2+1}+\ln\left|\sqrt{x^2+1}+x\right|\right)+C.}
\end{aligned}$$

(i) $\int x^2\sqrt{4-x^2}\,dx$

Use $x = 2\sin\theta, \qquad dx = 2\cos\theta\,d\theta, \qquad 1-\sin^2\theta = \cos^2\theta$

$$\int x^2\sqrt{4-x^2}\,dx = \int 4\sin^2\theta\left(\sqrt{4(1-\sin^2\theta)}\right)\cdot(2\cos\theta\,d\theta) \quad = \quad 16\int \sin^2\theta\,\cos^2\theta\,d\theta$$

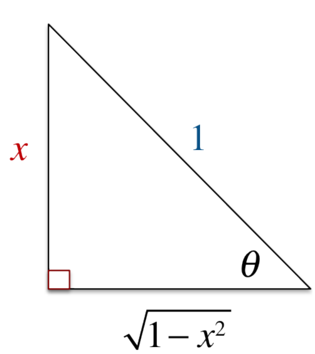
Use $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$ and $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$

$$\begin{aligned}
16\int \sin^2\theta\,\cos^2\theta\,d\theta &= 16\int\left(\frac{1-\cos(2\theta)}{2}\right)\cdot\left(\frac{1+\cos(2\theta)}{2}\right)\,d\theta\\
&= 4\int\left(1-\cos^2(2\theta)\right)\,d\theta = 4\int\left(1-\left(\frac{1+\cos(4\theta)}{2}\right)\right)\,d\theta\\
&= 2\int\left(1-\cos(4\theta)\right)\,d\theta\\
&= 2\left(\theta-\frac{1}{4}\sin(4\theta)\right)+C\\
&= 2\theta-\frac{1}{2}\sin(4\theta)+C = 2\theta-\frac{1}{2}\left(2\sin(2\theta)\cos(2\theta)\right)+C\\
&= 2\theta-\sin(2\theta)\cos(2\theta)+C = 2\theta-(2\sin\theta\cos\theta)\left(\cos^2\theta-\sin^2\theta\right)+C\\
&= 2\theta-2\sin\theta\cos^3\theta+2\sin^3\theta\cos\theta+C\\
&= 2\sin^{-1}\theta-2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)^3+2\left(\frac{x}{2}\right)^3\left(\frac{\sqrt{4-x^2}}{2}\right)+C\\
&= \boxed{2\sin^{-1}x-\frac{1}{8}x(4-x^2)^{3/2}+\frac{1}{8}x^3\sqrt{4-x^2}+C.}
\end{aligned}$$

(j) $\int \dfrac{x^2}{\sqrt{1-x^2}}\,dx$

Use $x = \sin\theta, \qquad dx = \cos\theta\,d\theta, \qquad 1-\sin^2\theta = \cos^2\theta$

$$\int\frac{x^2}{\sqrt{1-x^2}}\,dx = \int\frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}(\cos\theta\,d\theta) \quad = \quad \int\sin^2\theta\,d\theta$$



$$\begin{aligned}
&= \int\frac{1-\cos(2\theta)}{2}\,d\theta\\
&= \frac{1}{2}\theta-\frac{1}{4}\sin(2\theta)+C = \frac{1}{2}\theta-\frac{1}{4}(2\sin\theta\cos\theta)+C\\
&= \frac{1}{2}\theta-\frac{1}{2}\sin\theta\cos\theta+C\\
&= \frac{1}{2}\sin^{-1}x-\frac{1}{2}(x)\left(\sqrt{1-x^2}\right)+C\\
&= \boxed{\frac{1}{2}\sin^{-1}x-\frac{1}{2}x\sqrt{1-x^2}+C}
\end{aligned}$$

(k) $\displaystyle\int \frac{x^2}{\sqrt{4+x^2}}\,dx$

Use $x = 2\tan\theta, \qquad dx = 2\sec^2\theta\,d\theta, \qquad 1+\tan^2\theta = \sec^2\theta$

$$\int \frac{x^2}{\sqrt{4+x^2}}\,dx = \int \frac{4\tan^2\theta}{\sqrt{4\left(1+\tan^2\theta\right)}}\cdot\left(2\sec^2\theta\,d\theta\right) = 4\int \tan^2\theta\sec\theta\,d\theta$$
$$= 4\int\left(\sec^2\theta-1\right)\sec\theta\,d\theta = 4\int\left(\sec^3\theta-\sec\theta\right)\,d\theta$$

Recall from problem 8 that

$$\int \sec^3\theta\,d\theta = \frac{1}{2}\left(\sec\theta\,\tan\theta+\int\sec\theta\,d\theta\right)$$

then the solution is

$$\begin{aligned}
\int \frac{x^2}{\sqrt{4+x^2}}\,dx &= 4\int\sec^3\theta\,d\theta-4\int\sec\theta\,d\theta\\
&= 4\cdot\frac{1}{2}\left(\sec\theta\,\tan\theta+\int\sec\theta\,d\theta\right)-4\int\sec\theta\,d\theta\\
&= 2\sec\theta\,\tan\theta+2\int\sec\theta\,d\theta-4\int\sec\theta\,d\theta\\
&= 2\sec\theta\,\tan\theta-2\int\sec\theta\,d\theta+C\\
&= 2\sec\theta\,\tan\theta-2\ln|\sec\theta+\tan\theta|+C\\
&= 2\left(\frac{\sqrt{x^2+4}}{2}\right)\left(\frac{x}{2}\right)-2\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right|+C\\
&= \boxed{\frac{1}{2}x\sqrt{x^2+4}-2\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right|+C.}
\end{aligned}$$

(l) $\displaystyle\int \frac{3x-1}{x^2+1}\,dx$

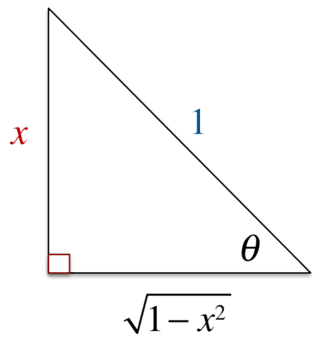
Use $x = \tan\theta, \qquad dx = \sec^2\theta\,d\theta, \qquad 1+\tan^2\theta = \sec^2\theta$

$$\begin{aligned}
\int \frac{3x-1}{x^2+1}\,dx = \int\frac{3\tan\theta-1}{1+\tan^2\theta}\left(\sec^2\theta\,d\theta\right) &= \int\frac{3\tan\theta-1}{\sec^2\theta}\left(\sec^2\theta\,d\theta\right)\\
&= \int\left(3\tan\theta-1\right)\,d\theta\\
&= \int\left(3\frac{\sin\theta}{\cos\theta}-1\right)\,d\theta = 3\int\frac{\sin\theta}{\cos\theta}\,d\theta-\int d\theta\\
&= 3\int\frac{1}{u}\,(-du)-\int d\theta\\
&= -3\int\frac{1}{u}\,du-\int d\theta\\
&= -3\ln|u|-\theta+C = -3\ln|\cos\theta|-\theta+C\\
&= \boxed{-3\ln\left|\frac{1}{\sqrt{x^2+1}}\right|-\tan^{-1}x+C.}
\end{aligned}$$

(m) $\displaystyle\int \frac{(1-x^2)^{3/2}}{x^2}\,dx$

Use $x = \sin\theta, \quad dx = \cos\theta\,d\theta, \quad 1-\sin^2\theta = \cos^2\theta$

$$\int \frac{(1-x^2)^{3/2}}{x^2}\,dx = \int \frac{(1-\sin^2\theta)^{3/2}}{\sin^2\theta}(\cos\theta\,d\theta) = \int \frac{(\cos^2\theta)^{3/2}}{\sin^2\theta}(\cos\theta\,d\theta)$$
$$= \int \frac{\cos^3\theta}{\sin^2\theta}(\cos\theta\,d\theta) = \int \frac{\cos^4\theta}{\sin^2\theta}\,d\theta$$
$$= \int \frac{\left(1-\sin^2\theta\right)^2}{\sin^2\theta}\,d\theta = \int \frac{1-2\sin^2\theta+\sin^4\theta}{\sin^2\theta}\,d\theta$$



$$= \int \frac{1}{\sin^2\theta}\,d\theta - 2\int d\theta + \int \sin^2\theta\,d\theta$$
$$= \int \csc^2\theta\,d\theta - 2\int d\theta + \int \frac{1-\cos(2\theta)}{2}\,d\theta$$
$$= -\cot\theta \quad -2\theta \quad + \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) + C$$
$$= -\cot\theta \quad -2\theta \quad + \frac{1}{2}\theta - \frac{1}{4}(2\sin\theta\cos\theta) + C$$
$$= -\frac{\sqrt{1-x^2}}{x} - \frac{3}{2}\sin^{-1}\theta - \frac{1}{2}(x)\left(\sqrt{1-x^2}\right) + C$$
$$= \boxed{-\frac{\sqrt{1-x^2}}{x} - \frac{3}{2}\sin^{-1}x - \frac{x}{2}\sqrt{1-x^2} + C.}$$

(n) $\displaystyle\int \frac{\sqrt{1-x^2}}{x}\,dx$

Use $x = \sin\theta, \quad dx = \cos\theta\,d\theta, \quad 1-\sin^2\theta = \cos^2\theta$

$$\int \frac{\sqrt{1-x^2}}{x}\,dx = \int \frac{\sqrt{1-\sin^2\theta}}{\sin\theta}(\cos\theta\,d\theta) = \int \frac{\cos^2\theta}{\sin\theta}\,d\theta$$
$$= \int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta = \int \frac{1}{\sin\theta}\,d\theta - \int \sin\theta\,d\theta$$
$$= \int \csc\theta\,d\theta - \int \sin\theta\,d\theta$$
$$= \ln|\csc\theta - \cot\theta| + \cos\theta + C$$
$$= \boxed{\ln\left|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C.}$$

If you don't remember the integral of $\csc\theta$:

$$\int \csc\theta\,d\theta = \int \frac{1}{\sin\theta}\,d\theta = \int \frac{\sin\theta}{\sin^2\theta}\,d\theta = \int \frac{\sin\theta}{1-\cos^2\theta}\,d\theta$$

and use the substitution $u = \cos\theta$, $du = -\sin\theta\,d\theta$.

(o) $\int \frac{1}{\sqrt{x^2-81}}\,dx$

Use $x = 9\sec\theta, \qquad dx = 9\sec\theta\tan\theta\,d\theta, \qquad \sec^2\theta - 1 = \tan^2\theta$

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-81}}\,dx &= \int \frac{1}{\sqrt{81\sec^2\theta-81}}\,[9\sec\theta\tan\theta\,d\theta] \\
&= \int \frac{1}{9\tan\theta}\,[9\sec\theta\tan\theta\,d\theta] \\
&= \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C \\
&= \ln\left|\frac{x}{9}+\frac{\sqrt{x^2-81}}{9}\right| + C.
\end{aligned}$$

(p) $\int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx$

Use $x = \tan\theta, \qquad dx = \sec^2\theta\,d\theta, \qquad \tan^2\theta + 1 = \sec^2\theta$

To evaluate the limits:

- When $x = 0$, we have $\tan\theta = 0$ which gives $\theta = 0$
- When $x = 1$, we have $\tan\theta = 1$ which gives $\theta = \frac{\pi}{4}$

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx &= \int_0^{\pi/4} \frac{1}{\sqrt{\tan^2\theta+1}}\,\left[\sec^2\theta\,d\theta\right] \\
&= \int_0^{\pi/4} \frac{1}{\sec\theta}\,\left[\sec^2\theta\,d\theta\right] = \int_0^{\pi/4} \sec\theta\,d\theta \\
&= \ln|\sec\theta+\tan\theta|\,\Big|_0^{\pi/4} \\
&= \ln\left|\sec\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{4}\right)\right| - \ln|\sec(0)+\tan(0)| \\
&= \ln\left|\sqrt{2}+1\right| - \ln|1+0| \\
&= \ln\left|\sqrt{2}+1\right|.
\end{aligned}$$

(q) $\int_0^5 \sqrt{10x - x^2}\, dx$

First, complete the square

$$10x - x^2 = -\left(x^2 - 10x + 25\right) + 25 = 25 - (x-5)^2 = 25\left[1 - \frac{(x-5)^2}{25}\right] = 25\left[1 - \left(\frac{x-5}{5}\right)^2\right]$$

The appropriate substitution is $\frac{x-5}{5} = \sin\theta$, with $dx = 5\cos\theta\, d\theta$, and the integral becomes The appropriate substitution is $\frac{x-5}{5} = \sin\theta$, with $dx = 5\cos\theta\, d\theta$, and the integral becomes

$$\begin{aligned}
\int \sqrt{10x - x^2}\, dx = \int \sqrt{25\left[1 - \left(\frac{x-5}{5}\right)^2\right]}\, dx &= \int \sqrt{25\left[1 - \sin^2\theta\right]}\, [5\cos\theta\, d\theta] \\
&= \int \sqrt{25\cos^2\theta}\, [5\cos\theta\, d\theta] \\
&= \int [5\cos\theta]\, [5\cos\theta\, d\theta] = 25\int \cos^2\theta\, d\theta \\
&= 25\int \frac{1+\cos(2\theta)}{2}\, d\theta \\
&= 25\left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right] + C \\
&= 25\left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\,\cos\theta\right] + C \\
&= \frac{25}{2}\left[\sin^{-1}\left(\frac{x-5}{5}\right) + \left(\frac{x-5}{5}\right)\cdot\left(\frac{\sqrt{25-(x-5)^2}}{5}\right)\right]
\end{aligned}$$

Here we solve the indefinite integral first and then evaluate the limits.

$$\begin{aligned}
\int_0^5 \sqrt{10x - x^2}\, dx &= \frac{25}{2}\left[\sin^{-1}\left(\frac{x-5}{5}\right)\Big|_0^5 + \frac{(x-5)\sqrt{25-(x-5)^2}}{25}\Big|_0^5\right] \\
&= \frac{25}{2}\left[\sin^{-1}(0) - \sin^{-1}\left(\frac{-5}{5}\right) + 0 - 0\right] \\
&= \frac{25}{2}\left[0 - \sin^{-1}(-1)\right] = \frac{25}{2}\left[-\frac{3\pi}{2}\right] \\
&= -\frac{75\pi}{4}
\end{aligned}$$