1. What are the three Pythagorean trigonometric identities?

The **only one** you need to remember is $\sin^2 x + \cos^2 x = 1.$

The other two can be found by dividing by $\sin^2 x$ or $\cos^2 x$.

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} &= \frac{1}{\sin^2 x} \\
1 + \cot^2 x &= \csc^2 x.
\end{aligned}$$

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\sin^2 x + \cos^2 x &= 1 \\
\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\
\tan^2 x + 1 &= \sec^2 x.
\end{aligned}$$

2. What are the power reduction (half angle) formulas for $\sin^2 x$ and $\cos^2 x$?

For the half-angle and double angle formulas we just need to remember the following two identities

$$\begin{aligned}
\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta
\end{aligned}$$

To find the double angle formulas use $\alpha = \beta$

$$\begin{aligned}
\sin(\alpha + \alpha) = \sin(2\alpha) &= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha = 2\sin\alpha\cos\alpha \\
\cos(\alpha + \alpha) = \cos(2\alpha) &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha = \cos^2\alpha - \sin^2\alpha
\end{aligned}$$

To solve for the quadratic form in terms of the double angle, substitute the Pythagorean identity

$$\begin{aligned}
\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\
&= \cos^2\alpha - (1 - \cos^2\alpha) = \cos^2\alpha - 1 + \cos^2\alpha = 2\cos^2\alpha - 1 \\
&\Rightarrow \\
2\cos^2\alpha &= 1 + \cos(2\alpha) \\
\cos^2\alpha &= \frac{1 + \cos(2\alpha)}{2}
\end{aligned}$$

$$\begin{aligned}
\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\
&= 1 - \sin^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha \\
&\Rightarrow \\
2\sin^2\alpha &= 1 - \cos(2\alpha) \\
\sin^2\alpha &= \frac{1 - \cos(2\alpha)}{2}
\end{aligned}$$

Finally, to obtain the half-angle formulas replace α for $\frac{\gamma}{2}$ in any of the equations above.

3. Evaluate the following integrals

(a). $\displaystyle\int \sin^4 x\,dx$

Use $\boxed{\sin^2 x = \dfrac{1-\cos(2x)}{2}}$

$$\int \sin^4 x\,dx = \int \left(\sin^2 x\right)^2 dx = \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx = \frac{1}{4}\int (1-\cos(2x))^2\,dx$$
$$= \frac{1}{4}\int \left(1-2\cos(2x)+\cos^2(2x)\right)\,dx$$

Use $\boxed{\cos^2(2x) = \dfrac{1+\cos(4x)}{2}}$

$$\int \sin^4 x\,dx = \frac{1}{4}\int \left(1-2\cos(2x)+\cos^2(2x)\right)\,dx = \frac{1}{4}\int \left(1-2\cos(2x)+\frac{1+\cos(4x)}{2}\right) dx$$
$$= \frac{1}{4}\left[x-\sin(2x)+\frac{x}{2}+\frac{1}{8}\sin(4x)\right]+C$$
$$= \boxed{\frac{3}{8}x-\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C.}$$

(b). $\displaystyle\int \frac{\sin x}{\cos^3 x}\,dx$

Use $\boxed{u=\cos x \;\Rightarrow\; du = -\sin x\,dx}$

$$\int \frac{\sin x}{\cos^3 x}\,dx = -\int \frac{1}{\cos^3 x}(-\sin x\,dx) = -\int \frac{1}{u^3}\,du$$
$$= -\left(-\frac{1}{2}u^{-2}\right)+C = \frac{1}{2}\frac{1}{u^2} + C$$
$$= \boxed{\frac{1}{2}\frac{1}{\cos^2 x}+C = \frac{1}{2}\sec^2 x + C.}$$

(c). $\displaystyle\int \cos^2 x\,dx$

Use $\boxed{\cos^2 x = \dfrac{1+\cos(2x)}{2}}$

$$\int \cos^2 x\,dx = \int \frac{1+\cos(2x)}{2}\,dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos(2x)\,dx$$
$$= \boxed{\frac{1}{2}x + \frac{1}{4}\sin(2x) + C.}$$

(d). $\int \sin^3 x \cos^2 x \, dx$

Use $u = \cos x \Rightarrow du = -\sin x \, dx$

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \, (\sin x \, dx) &= \int \left(1 - \cos^2 x\right) \cos^2 x \, (\sin x \, dx) \\
&= \int \left(\cos^2 x - \cos^4 x\right) (\sin x \, dx) \\
&= \int \left(u^2 - u^4\right)(-du) = \int \left(u^4 - u^2\right) du \\
&= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\
&= \boxed{\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.}
\end{aligned}$$

(e). $\int_0^{\pi/2} \sin^2 x \cos^2 x \, dx$

Use $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$ and $\sin^2 x = \dfrac{1 - \cos(2x)}{2}$

$$\begin{aligned}
\int_0^{\pi/2} \left(\frac{1 - \cos(2x)}{2}\right)\left(\frac{1 + \cos(2x)}{2}\right) dx &= \frac{1}{4}\int_0^{\pi/2} \left(1 - \cos^2(2x)\right) \, dx \\
&= \frac{1}{4}\int_0^{\pi/2} dx - \frac{1}{4}\int_0^{\pi/2} \cos^2(2x) \, dx \\
&= \frac{1}{4} \, x\Big|_0^{\pi/2} - \frac{1}{4}\left[\frac{1}{4}(2x) + \frac{1}{8}\sin(4x)\right]\Bigg|_0^{\pi/2} \\
&= \frac{\pi}{8} - \frac{1}{4}\left[\frac{\pi}{4} + \frac{1}{8}\left(\sin(2\pi) - \sin(0)\right)\right] \\
&= \frac{\pi}{8} - \frac{\pi}{16} = \boxed{\frac{\pi}{16}.}
\end{aligned}$$

(f). $\int \sin^3 x \, dx$

Use $u = \cos x \Rightarrow du = -\sin x \, dx$

$$\begin{aligned}
\int \sin^3 x \, dx = \int \sin^2 x \, (\sin x \, dx) = \int \left(1 - \cos^2 x\right)(\sin x \, dx) = \int \left(1 - u^2\right)(-du) &= \int \left(u^2 - 1\right) du \\
&= \frac{1}{3}u^3 - u + C \\
&= \boxed{\frac{1}{3}\cos^3 x - \cos x + C.}
\end{aligned}$$

(g). $\int x\sec^2(x^2)\tan^4(x^2)\,dx$

Use $u = x^2 \;\Rightarrow\; du = 2\,x\,dx$

$$\int x\sec^2(x^2)\tan^4(x^2)\,dx = \frac{1}{2}\int \sec^2(u)\tan^4(u)\,du$$

Use $v = \tan u \;\Rightarrow\; dv = \sec^2 u\,du$

$$\begin{aligned}\frac{1}{2}\int \sec^2(u)\tan^4(u)\,du = \frac{1}{2}\int v^4\,dv = \frac{1}{2}\left(\frac{1}{5}v^5\right)+C &= \frac{1}{10}v^5 + C\\ &= \frac{1}{10}\tan^5 u + C\\ &= \boxed{\frac{1}{10}\tan^5(x^2)+C.}\end{aligned}$$

(h). $\int \tan^2 x\,dx$

Use $\tan^2 x = \sec^2 x - 1$

$$\int \tan^2 x\,dx = \int \left(\sec^2 x - 1\right)\,dx = \boxed{\tan x - x + C.}$$

(i). $\int \tan^5 x\sec^3 x\,dx$

Use $\tan^2 x = \sec^2 x - 1$ and $\frac{d}{dx}(\sec(x)) = \sec x\,\tan x$

$$\begin{aligned}\int \tan^5 x\sec^3 x\,dx &= \int \tan^4 x\,\sec^2 x\,(\sec x\tan x\,dx)\\ &= \int \left(\tan^2 x\right)^2\sec^2 x\,(\sec x\tan x\,dx)\\ &= \int \left(\sec^2 x - 1\right)^2\sec^2 x\,(\sec x\tan x\,dx)\\ &= \int \left(\sec^4 x - 2\,\sec^2 x + 1\right)\sec^2 x\,(\sec x\tan x\,dx)\\ &= \int \left(\sec^6 x - 2\,\sec^4 x + \sec^2 x\right)(\sec x\tan x\,dx)\end{aligned}$$

Using the substitution $u = \sec x$, $du = \sec x\tan x\,dx$ we obtain

$$\begin{aligned}\int \tan^5 x\,\sec^3 x\,dx &= \int \left(u^6 - 2u^4 + u^2\right)\,du\\ &= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C\\ &= \boxed{\frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C.}\end{aligned}$$

(j). $\int \sin(8x)\cos(5x)\,dx$

With this type of integrals we integrate by parts *twice* and rearrange terms.

Use $u = \sin(8x), \quad du = 8\cos(8x)\,dx \quad$ and $\quad dv = \cos(5x)\,dx, \quad v = \frac{1}{5}\sin(5x)$

$$\int \sin(8x)\cos(5x)\,dx = \frac{1}{5}\sin(8x)\sin(5x) - \frac{8}{5}\int \sin(5x)\,\cos(8x)\,dx$$

Next use $u = \cos(8x), \quad du = -8\sin(8x)\,dx \quad$ and $\quad dv = \sin(5x)\,dx, \quad v = -\frac{1}{5}\cos(5x)$

$$\begin{aligned}\int \sin(8x)\cos(5x)\,dx &= \frac{1}{5}\sin(8x)\sin(5x) + \frac{8}{5}\int \sin(5x)\,\cos(8x)\,dx \\ &= \frac{1}{5}\sin(8x)\sin(5x) - \frac{8}{5}\left[-\frac{1}{5}\cos(8x)\,\cos(5x) - \frac{8}{5}\int \sin(8x)\,\cos(5x)\,dx\right] \\ &= \frac{1}{5}\sin(8x)\sin(5x) + \frac{8}{25}\cos(8x)\cos(5x) + \frac{64}{25}\int \sin(8x)\,\cos(5x)\,dx\end{aligned}$$

We subtract the last integral from both sides

$$\begin{aligned}\int \sin(8x)\,\cos(5x)\,dx - \frac{64}{25}\int \sin(8x)\,\cos(5x)\,dx &= \frac{1}{5}\sin(8x)\sin(5x) + \frac{8}{25}\cos(8x)\cos(5x) + \\ &\quad \frac{64}{25}\int \sin(8x)\,\cos(5x)\,dx - \frac{64}{25}\int \sin(8x)\,\cos(5x)\,dx\end{aligned}$$

to obtain

$$\begin{aligned}-\frac{39}{25}\int \sin(8x)\,\cos(5x)\,dx &= \frac{1}{5}\sin(8x)\sin(5x) + \frac{8}{25}\cos(8x)\cos(5x) \\ \int \sin(8x)\,\cos(5x)\,dx &= \boxed{-\frac{25}{39}\left[\frac{1}{5}\sin(8x)\sin(5x) + \frac{8}{25}\cos(8x)\cos(5x)\right] + C.}\end{aligned}$$

(k). $\int x^2 \sin x\,dx$

Use integration by parts and tabulate

Differentiate u		Integrate v
x^2		$\sin x$
$2x$	+	$-\cos x$
2	-	$-\sin x$
0	+	$\cos x$

$$\begin{aligned}\int x^2 \sin x\,dx &= (x^2)\cdot(-\cos x) - (2x)\cdot(-\sin x) + 2\cdot(\cos x) + C \\ &= \boxed{-x^2\cos x + 2\,x\sin x + 2\,\cos x + C.}\end{aligned}$$

(l). $\displaystyle\int \frac{1-\tan^2 x}{\sec^2 x}\,dx$

Use $\tan^2 x = \sec^2 x - 1$

$$\int \frac{1-\tan^2 x}{\sec^2 x}\,dx = \int \frac{1-\left(\sec^2 x - 1\right)}{\sec^2 x}\,dx = \int \frac{2-\sec^2 x}{\sec^2 x}\,dx = \int \frac{2}{\sec^2 x}\,dx - \int dx = \int 2\cos^2 x\,dx - \int dx$$

Use $\cos^2 x = \dfrac{1+\cos(2x)}{2}$

$$\int \frac{1-\tan^2 x}{\sec^2 x}\,dx = \int 2\cos^2 x\,dx - \int dx = \int 2\left(\frac{1+\cos(2x)}{2}\right)dx - \int dx = \int \left(1+\cos(2x)\right)\,dx - \int dx = x + \frac{1}{2}\sin(2x) - x + C = \boxed{\frac{1}{2}\sin(2x) + C.}$$

(m). $\displaystyle\int_{\pi/4}^{\pi/2} \cot^3 x\,dx$

Use $\cot^2 x = \csc^2 x - 1$

$$\int_{\pi/4}^{\pi/2} \cot^3 x\,dx = \int_{\pi/4}^{\pi/2} \cot^2 x\,\cot x\,dx = \int_{\pi/4}^{\pi/2} \left(\csc^2 x - 1\right)\cot x\,dx$$

We separate this integral into two integrals:

- For the first one we use $u = \cot x$, $du = -\csc^2 x\,dx$, when $x = \frac{\pi}{4}$, $u = \cot\left(\frac{\pi}{4}\right) = 1$ and when $x = \frac{\pi}{2}$, $u = \cot\left(\frac{\pi}{2}\right) = 0$

$$\int_{\pi/4}^{\pi/2} \cot x\,\left(\csc^2 x\,dx\right) = -\int_1^0 u\,du = \int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}.$$

- For the second one we use $\sin x$ and $\cos x$

$$-\int_{\pi/4}^{\pi/2} \cot x\,dx = -\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\,dx$$

and $u = \sin x$, $du = \cos x\,dx$, when $x = \frac{\pi}{4}$, $u = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and when $x = \frac{\pi}{2}$, $u = \sin\left(\frac{\pi}{2}\right) = 1$

$$-\int_{\pi/4}^{\pi/2} \cot x\,dx = -\int_{\sqrt{2}/2}^{1} \frac{1}{u}\,du = \int_1^{\sqrt{2}/2} \frac{1}{u}\,du = \ln(u)\big|_1^{\sqrt{2}/2} = \ln\left(\frac{\sqrt{2}}{2}\right) - \ln(1) = \ln\left(\frac{\sqrt{2}}{2}\right).$$

So that our solution is

$$\int_{\pi/4}^{\pi/2} \cot^3 x\,dx = \boxed{\frac{1}{2} + \ln\left(\frac{\sqrt{2}}{2}\right).}$$

(n). $\int \sec^2 x \, dx$

Use $\dfrac{d}{dx} \tan x = \sec^2 x$

$$\int \sec^2 x \, dx = \boxed{\tan x + C.}$$

(o). $\int \csc x \cot x \, dx$

Use $\dfrac{d}{dx} \csc x = -\csc x \cot x$

$$\int \csc x \cot x \, dx = \boxed{-\csc x + C.}$$