1. Find the Maclaurin series for $f(x)$ and the associated radius of convergence

(a) $f(x) = (1-x)^{-2}$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1-x)^{-2}$	1
1	$2(1-x)^{-3}$	2
2	$6(1-x)^{-4}$	6
3	$24(1-x)^{-5}$	24
4	$120(1-x)^{-6}$	120
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}(1-x)^{-2} &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\ &= 1 + 2x + \frac{6}{2}x^2 + \frac{24}{6}x^3 + \frac{120}{24}x^4 + \cdots \\ &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots \\ &= \sum_{n=0}^{\infty}(n+1)x^n\end{aligned}$$

To find the radius of convergence we use the Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = |x| \cdot \lim_{n\to\infty}\frac{n+2}{n+1} = |x|\cdot 1 = |x| < 1,$$

for convergence, so $R = 1$.

(b) $f(x) = \cos(3x)$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos(3x)$	1
1	$-3\sin(3x)$	0
2	$-3^2\cos(3x)$	-3^2
3	$3^3\sin(3x)$	0
4	$3^4\cos(3x)$	3^4
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\cos(3x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\ &= 1 - \frac{3^2}{2}x^2 + \frac{3^4}{24}x^4 + \cdots \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{3^{2n}}{(2n)!}x^{2n}\end{aligned}$$

To find the radius of convergence we use the Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{2n+2}x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{3^{2n}x^{2n}}\right| = 3^2x^2\cdot\lim_{n\to\infty}\frac{1}{(2n+1)(2n+1)} = 9x^2\cdot 0 = 0 < 1,$$

this hold for any x so that $R = \infty$.

(c) $f(x) = x\,e^x$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^x	0
1	$(x+1)e^x$	1
2	$(x+2)e^x$	2
3	$(x+3)e^x$	3
4	$(x+4)e^x$	4
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}x\,e^x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\ &= x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4 + \cdots \\ &= \sum_{n=0}^{\infty}\frac{n}{n!}x^n = \sum_{n=1}^{\infty}\frac{n}{n!}x^n = \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!}\end{aligned}$$

To find the radius of convergence we use the Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n!}\cdot\frac{(n-1)!}{x^n}\right| = |x|\cdot\lim_{n\to\infty}\frac{1}{n} = |x|\cdot 0 = 0 < 1,$$

this hold for any x so that $R = \infty$.

(d) $f(x) = e^{5x}$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{5x}	1
1	$5e^{5x}$	5
2	5^2e^{5x}	25
3	5^3e^{5x}	125
4	5^4e^{5x}	625
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} x\,e^x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\ &= 1 + 5x + \frac{25}{2!}x^2 + \frac{125}{3!}x^3 + \frac{625}{4!}x^4 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{n}{n!}x^n = \sum_{n=0}^{\infty} \frac{5^n}{n!}\,x^n. \end{aligned}$$

To find the radius of convergence we use the Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{5^{n+1}x^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n\,x^n} \right| = 5\,|x| \cdot \lim_{n\to\infty} \frac{1}{n+1} = 5\,|x| \cdot 0 = 0 < 1,$$

this hold for any x so that $R = \infty$.

2. Find the Taylor series for $f(x)$ centered at the given value of a.

(a) $f(x) = x^4 - 3x^2 + 1, \quad a = 1$

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^4 - 3x^2 + 1$	-1
1	$4x^3 - 6x$	-2
2	$12\,x^2 - 6$	6
3	$24\,x$	24
4	24	24
5	0	0
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} x^4 - 3x^2 + 1 &= f(1) + f'(1)x + \frac{f''(1)}{2!}x^2 + \frac{f'''(1)}{3!}x^3 + \frac{f^{(4)}(1)}{4!}. \\ &= \frac{-1}{0!}(x-1)^0 + \frac{-2}{1!}(x-1)^1 + \frac{6}{2!}(x-1)^2 \\ &\quad + \frac{24}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4 \\ &= -1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4. \end{aligned}$$

Since $f^{(n)}(x) = 0$ for $n \geq 5$, we say $f(x)$ has a finite series expansion about $a = 1$. and we don't have to perform a convergence test, because finite series converge for any value of x, so that $R = \infty$.

(b) $f(x) = x^{-2}, \quad a = 1$

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6
3	$-24x^{-5}$	-24
4	$120x^{-6}$	120
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} x^{-2} &= f(1) + f'(1)\,x + \frac{f''(1)}{2!}x^2 + \frac{f'''(1)}{3!}x^3 + \frac{f^{(4)}(1)}{4!}x^4 + \cdots \\ &= 1 - 2(x-1) + 6\cdot\frac{(x-1)^2}{2!} - 24\cdot\frac{(x-1)^3}{3!} + 120\cdot\frac{(x-1)^4}{4!} + \cdots \\ &= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n. \end{aligned}$$

To find the radius of convergence we use the Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(x-1)^{n+1}}{(n+1)(x-1)^n}\right| = |x-1| \cdot \lim_{n\to\infty} \frac{n+2}{n+1} = |x-1|\cdot 1 = |x-1| < 1,$$

for convergence. So the series converges for $|x-1| < 1$ and $R = 1$.

(c) $f(x) = \cos x, \quad a = \pi$

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\cos x$	-1
1	$-\sin x$	0
2	$-\cos x$	1
3	$\sin x$	0
4	$\cos x$	-1
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} \cos x &= f(\pi) + f'(\pi)\,x + \frac{f''(\pi)}{2!}x^2 + \frac{f'''(\pi)}{3!}x^3 + \frac{f^{(4)}(\pi)}{4!}x^4 + \cdots \\ &= -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi)^{2n}}{(2n)!}. \end{aligned}$$

To find the radius of convergence we use the Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-\pi)^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(x-\pi)^{2n}}\right| = (x-\pi)^2 \cdot \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = (x-\pi)^2 \cdot 0 = 0 < 1,$$

for convergence. So the series converges for all x and $R = \infty$.

(d) $f(x) = \dfrac{1}{x}, \quad a = -3$

n	$f^{(n)}(x)$	$f^{(n)}(-3)$
0	$1/x$	$-1/3$
1	$-1/x^2$	$-1/3^2$
2	$2/x^3$	$-2/3^3$
3	$-6/x^4$	$-6/3^4$
4	$24/x^5$	$-24/3^5$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
\frac{1}{x} &= f(-3) + f'(-3)\,x + \frac{f''(-3)}{2!}x^2 + \frac{f'''(-3)}{3!}x^3 + \frac{f^{(4)}(-3)}{4!}x^4 + \cdots \\
&= \frac{-1/3}{0!}(x+3)^0 + \frac{-1/3^2}{1!}(x+3)^1 + \frac{-2/3^3}{2!}(x+3)^2 \\
&\quad + \frac{-6/3^4}{3!}(x+3)^3 + \frac{-24/3^5}{4!}(x+3)^4 + \cdots \\
&= -\frac{1}{3} - \frac{1}{3^2}(x+3) - \frac{1}{3^3}(x+3)^3 - \frac{1}{3^4}(x+3)^4 - \frac{1}{3^5}(x+3)^5 + \cdots \\
&= \sum_{n=0}^{\infty} -\frac{(x+3)^n}{3^{n+1}}.
\end{aligned}$$

To find the radius of convergence we use the Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+3)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{(x+3)^n}\right| = \lim_{n\to\infty}\left|\frac{x+3}{3}\right| = \frac{|x+3|}{3} < 1,$$

for convergence. So the series converges for $|x+3| < 3$ and $R = 3$.

3. Use series to approximate the definite integral.

(a) $\displaystyle\int_0^1 x\cos(x^3)\,dx$

First we find the Maclaurin series for $f(x)=\cos x$:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos(x)$	1
1	$-\sin(x)$	0
2	$-\cos(x)$	-1
3	$\sin(x)$	0
4	$\cos(x)$	1
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\cos(x) &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\cdots\\ &= 1-\frac{1}{2}x^2+\frac{1}{24}x^4+\cdots\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\end{aligned}$$

Then

$$\cos(x^3) = \sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n}}{(2n)!}.$$

And

$$x\cos(x^3) = x\left[\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n}}{(2n)!}\right]=\sum_{n=0}^{\infty}(-1)^n x\cdot\frac{x^{6n}}{(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+1}}{(2n)!}.$$

Note that all these series converge for all values of x, so that $R=\infty$.

The integral then becomes

$$\begin{aligned}\int_0^1 x\cos(x^3)\,dx=\int_0^1\left[\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+1}}{(2n)!}\right]dx &= \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(2n)!}\left[\int_0^1 x^{6n+1}\,dx\right]\right)\\ &= \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(2n)!}\left[\frac{x^{6n+2}}{6n+2}\right]_0^1\right)\\ &= \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(2n)!}\left[\frac{1^{6n+2}}{6n+2}-\frac{0^{6n+2}}{6n+2}\right]\right)\\ &= \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(2n)!}\left[\frac{1}{6n+2}+0\right]\right)=\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(2n)!}\left[\frac{1}{6n+2}\right]\right)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(6n+2)\,(2n)!}.\end{aligned}$$

(b) $\displaystyle\int_0^1 x^2\, e^{-x^2}\, dx$

First we find the Maclaurin series for $f(x) = e^x$:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^x	1
1	e^x	1
2	e^x	1
3	e^x	1
4	e^x	1
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} e^x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

Then

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

And

$$x^2\, e^{-x^2} = x^2 \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^2 \cdot x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+2}.$$

Note that all these series converge for all values of x, so that $R = \infty$.

The integral then becomes

$$\begin{aligned} \int_0^1 x^2\, e^{-x^2}\, dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+2} \right] dx &= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \left[\int_0^1 x^{2n+2}\, dx \right] \right) \\ &= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \left[\frac{x^{2n+3}}{2n+3} \right]_0^1 \right) \\ &= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \left[\frac{1^{2n+3}}{2n+3} - \frac{0^{2n+3}}{2n+3} \right] \right) \\ &= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \left[\frac{1}{2n+3} + 0 \right] \right) = \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \left[\frac{1}{2n+3} \right] \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+3)\, n!}. \end{aligned}$$

4. Use series to evaluate the limit

(a) $\lim_{x\to 0} \dfrac{x - \tan^{-1} x}{x^3}$

We need to express the numerator as a series and use the fact that power series are continuous functions

$$\begin{aligned}
\lim_{x\to 0} \frac{x - \tan^{-1} x}{x^3} &= \lim_{x\to 0} \frac{x - \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 - \cdots\right)}{x^3} \\
&= \lim_{x\to 0} \frac{\frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{7}x^7 - \cdots}{x^3} \\
&= \lim_{x\to 0} \left(\frac{1}{3} - \frac{1}{5}x^2 + \frac{1}{7}x^4 - \cdots\right) = \frac{1}{3}.
\end{aligned}$$

(b) $\lim_{x\to 0} \dfrac{\sin x - x + \frac{1}{6}x^3}{x^5}$

We need to express the numerator as a series and use the fact that power series are continuous functions

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} &= \lim_{x\to 0} \frac{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) - x + \frac{1}{6}x^3}{x^5} \\
&= \lim_{x\to 0} \frac{\frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x^5} \\
&= \lim_{x\to 0} \left(\frac{1}{5!} - \frac{1}{7!}x^2 + \frac{1}{9!}x^4 - \cdots\right) = \frac{1}{5!} = \frac{1}{120}.
\end{aligned}$$

5. Use Taylor's Inequality to determine the number of terms of the Maclaurin series for $f(x) = e^x$ that should be used to estimate $e^{0.1}$ to within 0.00001 error.

Taylor's inequality states: If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x - a| \leq d$, then the remainder, $R_n(x)$, of the Taylor series satisfies the inewuality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

Since all the derivatives of $f(x) = e^x$ are equal to e^x, the inequality becomes

$$|R_n(x)| \leq \frac{e^x}{(n+1)!}|x|^{n+1},$$

where $0 < x < 1$. Letting $x = 0.1$ we obtain

$$|R_n(0.1)| \leq \frac{e^{0.1}}{(n+1)!}|0.1|^{n+1} < 0.00001$$

Next we calculate the $Rn's$ to find the value of n that satisfies the last inequality

$$\begin{aligned}
R_0(0.1) &= .1105170918 \\
R_1(0.1) &= 5.525854590\,10^{-1} = 0.552854 \\
R_2(0.1) &= 1.841951530\,10^{-3} = 0.001841 \\
R_3(0.1) &= 4.604878825\,10^{-6} = 0.000004 \\
R_4(0.1) &= 9.209757650\,10^{-8} = 0.00000009
\end{aligned}$$

Then the inequality is satisfied by $n \geq 3$. This means that by adding the terms of the Maclaurin series for e^x corresponding to $n = 0, 1, 2$ and 3 we can estimate $e^{0.1}$ to within 0.00001 error.

6. A car is moving with speed 20 m/s and acceleration 2 m/s^2 at a given instant. Using a second-degree Taylor polynomial, estimate how far the car moves in the next second. Would it be reasonable to use this polynomial to estimate the distance traveled during the next minute?

Let $s(t)$ be the position function of the car, and for convenience set $s(0) = 0$. The velocity of the car is $v(t) = s'(t)$ and the acceleration is $a(t) = s''(t)$, so the second degree Taylor polynomial is

$$\begin{aligned}
P_2(t) &= s(0) + \frac{s'(0)}{1!}\,t + \frac{s''(0)}{2!}\,t^2 \\
&= s(0) + v(0)t + \frac{a(0)}{2}t^2 = 20\,t + t^2.
\end{aligned}$$

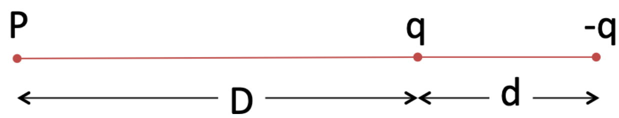
We estimate the distance traveled during the next second to be $s(1) \approx P_2(1) = 20(1) + 1^2 = 21$ m.

The function $P_2(t)$ would not be accurate over a full minute, since the car could not possible maintain an accelartion of 2 m/s^2 for that long. If it did, its final speed would be 140 m/s $\approx$ 313 mi/h.

7. An electric dipole consists of two electric charges of equal magnitude and opposite sign. If the charges are q and $-q$ and are located at a distance d from each other, then the electric field E at the point P in the figure is

$$E = \frac{q}{D^2} - \frac{q}{(D+d)^2}.$$

By expanding this expression for E as a series in powers of d/D, show that E is approximately proportional to $1/D^3$ when P is far away from the dipole.



In order to expand the equation in terms of d/D we need to reorganize some terms:

$$E = \frac{q}{D^2} - \frac{q}{(D-d)^2} = \frac{q}{D^2} - \frac{q}{D^2(1+d/D)^2} = \frac{q}{D^2}\left[1 - \left(1 + \frac{d}{D}\right)^{-2}\right]$$

And we need to expand the function $f(x) = (1+x)^{-2}$, for this we use Maclaurin series and tabulate the derivates as before,

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-2}$	1
1	$-2(1+x)^{-3}$	-2
2	$6(1+x)^{-4}$	6
3	$-24(1+x)x^{-5}$	-24
4	$120(1+x)^{-6}$	120
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}(1+x)^{-2} &= f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\ &= 1 - 2\,x + \frac{6}{2!}\,x^2 - \frac{24}{3!}\,x^3 + \frac{120}{4!}\,x^4 + \cdots \\ &= 1 - 2\,x + 3\,x^2 - 4\,x^3 + 5\,x^4 + \cdots\end{aligned}$$

Then our equation for the energy becomes

$$\begin{aligned}E = \frac{q}{D^2}\left[1 - \left(1 + \frac{d}{D}\right)^{-2}\right] &= \frac{q}{D^2}\left[1 - \left(1 - 2\,\frac{d}{D} + 3\left(\frac{d}{D}\right)^2 - 4\left(\frac{d}{D}\right)^3 + 5\left(\frac{d}{D}\right)^4 + \cdots\right)\right] \\ &= \frac{q}{D^2}\left[1 - 1 + 2\,\frac{d}{D} - 3\left(\frac{d}{D}\right)^2 + 4\left(\frac{d}{D}\right)^3 - 5\left(\frac{d}{D}\right)^4 + \cdots\right] \\ &= \frac{q}{D^2}\left[2\,\frac{d}{D} - 3\left(\frac{d}{D}\right)^2 + 4\left(\frac{d}{D}\right)^3 - 5\left(\frac{d}{D}\right)^4 + \cdots\right]\end{aligned}$$

When P is far away from the dipole $D >> d$, other words $d/D << 1$ or $d/D \to 0$ and if we only take the first term of the series we get

$$E \approx \frac{q}{D^2}\left[2\,\frac{d}{D}\right] = \frac{2\,q\,d}{D^3} = 2\,q\,d \cdot \frac{1}{D^3} \sim \frac{1}{D^3}.$$

8. If a surveyor measures differences in elevation when making plans for a highway across a desert, corrections must be made for the curvature of the earth. If the radius of the earth is R and L is the length of the highway, it can be shown that the correction is given by

$$C = R\sec\left(\frac{L}{R}\right) - R$$

Use a Taylor polynomial to show that

$$C \approx \frac{L^2}{2R} + \frac{5L^4}{24R^3}.$$

As before we reorganize terms,

$$C = R\left[\sec\left(\frac{L}{R}\right) - 1\right]$$

then the function to expand is $f(x) = \sec(x)$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sec(x)$	1
1	$\sec(x)\,\tan(x)$	0
2	$\sec(x)\,\big(2\sec^2(x) - 1\big)$	1
3	$\sec(x)\,\tan(x)\,\big(6\sec^2(x) - 1\big)$	0
4	$\sec(x)\,\big(18\sec^2(x)\tan^2(x) + 6\sec^4(x) - \sec^2(x) - \tan^2(x)\big)$	5
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
\sec(x) &\approx P_4(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
&= 1 + 0 + \frac{1}{2!}x^2 + 0 + \frac{5}{4!}x^4 = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4
\end{aligned}$$

And the equation for the correction becomes

$$\begin{aligned}
C = R\left[\sec\left(\frac{L}{R}\right) - 1\right] &\approx R\left[1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 - 1\right] \\
&= R\left[1 + \frac{1}{2}\left(\frac{L}{R}\right)^2 + \frac{5}{24}\left(\frac{L}{R}\right)^4 - 1\right] \\
&= R\left[\frac{1}{2}\left(\frac{L}{R}\right)^2 + \frac{5}{24}\left(\frac{L}{R}\right)^4\right] \\
&= R\left[\frac{1}{2}\frac{L^2}{R^2} + \frac{5}{24}\frac{L^4}{R^4}\right] \\
&= \frac{1}{2}\frac{L^2}{R} + \frac{5}{24}\frac{L^4}{R^3}.
\end{aligned}$$