Determine whether the series is convergent or divergent

**1.** $\displaystyle\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$

Use comparison test

$$\frac{4+3^n}{2^n} > \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n, \qquad \text{for all } n \geq 1.$$

The geometric series $\sum \left(\frac{3}{2}\right)^n$ diverges, since $3/2 > 1$, so that our series diverges by the comparison test.

Divergent

**2.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$

Use alternating series divergence test

$$\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1} = \sum_{n=1}^{\infty} (-1)^n b_n \quad \Rightarrow \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n-1}{2n+1} = \lim_{n\to\infty} \frac{3-\frac{1}{n}}{2+\frac{1}{n}} = \frac{3}{2} \neq 0$$

Divergent

**3.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Use Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}\right| = \lim_{n\to\infty} \left|\frac{2^n\,(n+1)^2}{2^{n+1}\,n^2}\right| = \frac{1}{2} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = \frac{1}{2} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 = \frac{1}{2} < 1$$

Convergent

**4.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3^n}$

Use comparison test

$$\frac{1}{n+3^n} < \frac{1}{3^n} = \left(\frac{1}{3}\right)^n, \qquad \text{for all } n \geq 1.$$

The geometric series $\sum \left(\frac{1}{3}\right)^n$ converges, since $1/3 < 1$, so that our series converges by the comparison test.

Convergent

**5.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

Use Root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-2)^n}{n^n}\right|} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1$$

Convergent

**6.** $\displaystyle\sum_{n=1}^{\infty} n^2\, e^{-n^3}$

Use Integral test

Let $f(x) = x^2\, e^{-x^3}$. Then $f$ is continuous and positive in $[1, \infty)$, and $f'(x) = \dfrac{x(2-3x^3)}{e^{x^3}} < 0$ for $x \geq 1$, so $f$ is decreasing in $[1, \infty)$. We can apply the integral test,

$$\int_1^{\infty} x^2\, e^{-x^3}\, dx = \lim_{b\to\infty} \left[\int_1^b x^2\, e^{-x^3}\, dx\right] = \lim_{b\to\infty} \left[-\frac{1}{3}e^{-x^3}\right]_1^b = 0 - \left(-\frac{1}{3}e^{-1}\right) = \frac{1}{3e}.$$

Here we used substitution to solve the integral.

Since the integral converges, the series also converges.

Convergent

**7.** $\displaystyle\sum_{n=2}^{\infty} \sin n$

Use test of divergence

Since $\displaystyle\lim_{n\to\infty} \sin n$ does not exist, the series diverges.

Divergent

**8.** $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^2 e^{-n}$

Use Limit Comparison test

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2 e^{-n}}{e^{-n}} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 = 1 > 0$$

Since the series $\sum e^{-n}$ is a convergent geometric series with $|r| = \dfrac{1}{e} < 1$, the series $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^2 e^{-n}$ also converges.

Convergent

**9.** $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$

Use Alternating Series test

$$\sum_{n=2}^{\infty} (-1)^{n-1} a_n = \sum_{n=2}^{\infty} (-1)^{n+1} a_n = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$$

Since $a_n = \frac{1}{n \ln n}$, we need to check that is decreasing and the limit of $a_n$ when $n$ goes to infinity.

- Since $\ln x > 0$ for $x \geq 2$, the $a_n$'s are positive and decreasing
- $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$

Convergent

**10.** $\sum_{k=1}^{\infty} \frac{k+5}{5^k}$

Use Ratio test

$$\lim_{k\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{k\to\infty} \left|\frac{k+6}{5^{k+1}} \cdot \frac{5^k}{k+5}\right| = \frac{1}{5} \lim_{k\to\infty} \frac{k+6}{k+5} = \frac{1}{5} \cdot 1 = \frac{1}{5} < 1$$

Convergent

**11.** $\sum_{n=1}^{\infty} \left(-\frac{n}{5}\right)^n$

Use test of divergence of alternating series

Since $\lim_{n\to\infty} \left(\frac{n}{5}\right)^n = \infty, \quad \Rightarrow \quad \lim_{n\to\infty} \left(-\frac{n}{5}\right)^n$ does not exists

Divergent

**12.** $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$

Use alternating series test $\quad a_n = \frac{n^2}{n^3+4}$

- To check if it is decreasing we use the first derivative of a function

$$\left(\frac{x^2}{x^3+4}\right)' = \frac{(x^3+4)(2x) - x^2(3x^2)}{(x^3+4)^2} = \frac{x(2x^3+8-3x^3)}{(x^3+4)^2} = \frac{x(8-x^3)}{(x^3+4)^2} < 0 \text{ for } x > 2$$

- $\lim_{n\to} a_n = \lim_{n\to\infty} \frac{n^2}{n^3+4} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{4}{n^3}} = 0$

Convergent

**13.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

Use Absolute convergence and integral test

We check the convergence of $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^4}\right| = \sum_{n=1}^{\infty} \frac{1}{n^4}$ using the integral test

$$\int_1^{\infty} \frac{1}{x^4}\,dx = \lim_{b\to\infty} \left[\int_1^b \frac{1}{x^4}\,dx\right] = \lim_{b\to\infty} \left[-\frac{1}{3x^3}\right]_1^b = 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}$$

Since the integral converges, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges and our series converges by the absolute convergence test.

Convergent

**14.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

Use the Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n^2+2n+2}{5^{n+1}} \cdot \frac{5^n}{n^2+1}\right| = \frac{1}{5} \cdot \lim_{n\to\infty} \frac{n^2+2n+2}{n^2+1} = \frac{1}{5} \cdot \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}} = \frac{1}{5}\cdot 1 = \frac{1}{5} < 1$$

Convergent

**15.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$

Use Comparison test

$$\frac{n^2-1}{3n^4+1} < \frac{n^2}{3n^4+1} < \frac{n^2}{3n^4} = \frac{1}{3}\frac{1}{n^2}$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the Integral test (see problem 13), then by the comparison test, our series converges.

Convergent

**16.** $\displaystyle\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

Use the Root test

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{-2n}{n+1}\right)^{5n}\right|} = \lim_{n\to\infty} \frac{2^5 n^5}{(n+1)^5} = 32 \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^5} = 32 \lim_{n\to\infty} \frac{1}{(1+1/n)^5} = 32\cdot 1 = 32 > 1$$

Divergent

**17.** $\displaystyle\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$

Use Integral test test and comparison test

$$\frac{k \ln k}{(k+1)^3} < \frac{k \ln k}{k^3} = \frac{\ln k}{k^2}$$

We integrate the last expression using integration by parts

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left[\int_1^b \frac{\ln x}{x^2}\,dx\right] = \lim_{b\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b}\right] - (-0-1) = 1$$

Here we used L'Hopital's to evaluate the first term in the limit.

Since the integral converges, the series converges and by the comparison test our original series also converges.

Convergent

**18.** $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$

Use Limit comparison test with $a_n = \sqrt[n]{2}$ and $b_n = 1/n$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt[n]{2}}{\frac{1}{n}} = \lim_{x\to\infty} \frac{2^{1/x} - 1}{1/x} \stackrel{LH}{=} \lim_{x\to\infty} \frac{2^{1/x} \cdot \ln 2 \cdot (-1/x^2)}{-1/x^2} = \lim_{x\to\infty} (2^{1/x} \cdot \ln 2) = 1 \cdot \ln 2 = \ln 2 > 0$$

Since $\sum b_n$ diverges (integral test) so does $\sum a_n$ and since $\sum \sqrt[n]{2}$ diverges, the series $\sum(\sqrt[n]{2} - 1)$ diverges.

Divergent

**19.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

Use Root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{n+1}\right)^{n^2}\right|} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n^2/n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n} = \lim_{n\to\infty} e^{\ln\left(\frac{n}{n+1}\right)^n}$$

$$= \exp\left[\lim_{n\to\infty} n \cdot \ln\left(\frac{n}{n+1}\right)\right] \stackrel{LH}{=} \exp[-1] = \frac{1}{e} < 1$$

Convergent

**20.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

Use Root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n^2+1}{2n^2+1}\right)^n\right|} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} = \frac{1}{2} < 1$$

Convergent

**21.** $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$

Use Comparison test

$$\frac{\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$$

Since the series $\sum \frac{1}{\sqrt{n}}$ diverges (Integral test), our series diverges by the Comparison test.

Divergent

**22.** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

Use Root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(2n+1)^n}{n^{2n}}\right|} = \lim_{n\to\infty} \frac{2n+1}{n^2} = \lim_{n\to\infty} \frac{\frac{2}{n}+\frac{1}{n^2}}{1} = 0 < 1$$

Convergent

**23.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} \sin\left(\frac{n\,\pi}{2}\right)$

Use Absolute convergence and integral tests

Note that $\sin\left(\frac{n\,\pi}{2}\right) = 0$ if $n$ is even and $(-1)^k$ if $n = 2k+1$, so that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \sin\left(\frac{n\,\pi}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^4} \cdot (-1)^n$$

Since the series $\sum \frac{1}{(2n+1)^4}$ converges by the integral test, our alternating series $\sum \frac{(-1)^n}{(2n+1)^4}$ converges by the absolute convergence test.

Convergent

**24.** $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(n+1)\,4^{2n+1}}$

Use the Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{10^{n+1}}{(n+2)4^{2n+3}} \cdot \frac{(n+1)4^{2n+1}}{10^n}\right| = \lim_{n\to\infty} \left(\frac{10}{4^2} \cdot \frac{n+1}{n+2}\right) = \frac{5}{8} < 1$$

Convergent

**25.** $\displaystyle\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$

Use the Limit comparison test with $a_n = \dfrac{n+2}{(n+1)^3}$ and $b_n = \dfrac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+2}{(n+1)^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2(n+2)}{(n+1)^3} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{\left(1+\frac{1}{n}\right)^3} = 1 > 0$$

Since $\sum \dfrac{1}{n^2}$ is a convergent series (Integral test), our series also converges.

Convergent

**26.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$

Use Alternating series test

We need to check two things:

- To check if it is decreasing we use the first derivative

$$\left(\frac{x}{x^2+2}\right)' = \frac{(x^2+2)(1) - x(2x)}{(x^2+2)^2} = \frac{2-x^2}{(x^2+2)^2} < 0 \text{ for } x \geq \sqrt{2}$$

- Check if the limit is equal to zero

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+2} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{2}{n}} = 0$$

Convergent