Determine whether the series is convergent or divergent

1. $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$

Use integral test

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\left[\int_0^b \frac{1}{1+x^2}\,dx\right]$$

Use trigonometric substitution to solve the integral

$$\lim_{b\to\infty}\left[\int_0^b \frac{1}{1+x^2}\,dx\right] = \lim_{b\to\infty}\left[\tan^{-1}x\right]_0^b = \lim_{b\to\infty}\left[\tan^{-1}b\right] - \tan^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

Since the improper integral converges, the series converges.

2. $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$

Geometric series

$$\sum_{n=0}^{\infty}(-1)^n\frac{5}{4^n} = \sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n\cdot 5 = \sum_{n=0}^{\infty} -\frac{5}{4}\left(-\frac{1}{4}\right)^{n-1} = -\left[\sum_{n=0}^{\infty}\frac{5}{4}\left(-\frac{1}{4}\right)^{n-1}\right] = -\left[\frac{5}{4}\left(-\frac{1}{4}\right)^{-1} + \sum_{n=1}^{\infty}\frac{5}{4}\left(-\frac{1}{4}\right)^{n-1}\right]$$

The series $\sum_{n=1}^{\infty}\frac{5}{4}\left(-\frac{1}{4}\right)^{n-1}$ is a geometric series with $a = 5/4$ and $r = -1/4$, since $|r| < 1$, it converges to

$$\sum_{n=1}^{\infty}\frac{5}{4}\left(-\frac{1}{4}\right)^{n-1} = \frac{a}{1-r} = \frac{\frac{5}{4}}{1-\left(-\frac{1}{4}\right)} = \frac{\frac{5}{4}}{\frac{5}{4}} = 1$$

Then our series converges to $-[-5+1] = 4$

3. $\sum_{n=2}^{\infty} \frac{n}{(n^2+2)^{3/2}}$

Use integral test

$$\int_2^{\infty}\frac{x}{(x^2+2)^{3/2}}\,dx = \lim_{b\to\infty}\left[\int_2^b \frac{x}{(x^2+2)^{3/2}}\,dx\right]$$

To solve the integral use the substitution $u = x^2 + 2$

$$\begin{aligned}\lim_{b\to\infty}\left[\int_2^b \frac{x}{(x^2+2)^{3/2}}\,dx\right] &= \lim_{b\to\infty}\left[-\frac{1}{\sqrt{x^2+2}}\right]_2^b = \lim_{b\to\infty}\left[-\frac{1}{\sqrt{b^2+2}}\right] - \left(-\frac{1}{\sqrt{2^2+2}}\right)\\ &= 0 + \frac{1}{\sqrt{6}} = \frac{1}{\sqrt{6}}\end{aligned}$$

Since the improper integral converges, the series converges.

4. $\sum_{n=0}^{\infty} \frac{1}{n+4}$

Use integral test

$$\int_0^{\infty} \frac{1}{x+4}\,dx = \lim_{b\to\infty}\left[\int_0^b \frac{1}{x+4}\,dx\right] = \lim_{b\to\infty}\left[\ln|x+4|\right]_0^b = \lim_{b\to\infty} ln\,|b+4| - \ln(0+4) = \infty - \ln(4) = \infty.$$

The series diverges.

5. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

Use integral test

$$\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{b\to\infty}\left[\int_1^b x^2 e^{-x^3}\,dx\right]$$

To solve the integral use the substitution $u = x^3$

$$\lim_{b\to\infty}\left[\int_1^b x^2 e^{-x^3}\,dx\right] = \lim_{b\to\infty}\left[-\frac{e^{-x^3}}{3}\right] = -\lim_{b\to\infty}\frac{e^{-b^3}}{3} + \frac{e^{-1}}{3} = 0 + \frac{1}{3e^1} = \frac{1}{3e^1}.$$

Since the improper integral converges, the series converges.

6. $\sum_{n=1}^{\infty} \ln\left(\frac{1}{3^n}\right)$

Use nth term test

$$\lim_{n\to\infty} \ln\left(\frac{1}{3^n}\right) = \ln\left(\lim_{n\to\infty}\frac{1}{3^n}\right) = \ln(0) = -\infty$$

Since the limit is not equal to zero, the series diverges.

7. $\sum_{n=2}^{\infty} \frac{1}{n\,(\ln n)^2}$

Use integral test

$$\int_2^{\infty} \frac{1}{x\,(\ln x)^2}\,dx = \lim_{b\to\infty}\left[\int_2^b \frac{1}{x\,(\ln x)^2}\,dx\right]$$

To solve the integral use the substitution $u = \ln(x)$

$$\lim_{b\to\infty}\left[\int_2^b \frac{1}{x\,(\ln x)^2}\,dx\right] = \lim_{b\to\infty}\left[-\frac{1}{\ln(x)}\right]_2^b = -\lim_{b\to\infty}\frac{1}{\ln(b)} + \frac{1}{\ln(2)} = 0 + \frac{1}{\ln(2)} = \frac{1}{\ln(2)}$$

Since the improper integral converges, the series converges.

8. $\sum_{n=1}^{\infty} n \tan \frac{1}{n}$

Use nth term test

$$\lim_{n\to\infty} n \tan \frac{1}{n} = \infty \cdot 0$$

Use L'Hopital's rule to evaluate the limit

$$\lim_{n\to\infty} n \tan \frac{1}{n} = \lim_{n\to\infty} \frac{\tan \frac{1}{n}}{\frac{1}{n}} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{-\frac{1+\tan^2 \frac{1}{x}}{x^2}}{-\frac{1}{x^2}}$$
$$= \lim_{n\to\infty} 1 + \tan^2 \frac{1}{x} = 1 + 0$$
$$= 1$$

Since the limit is different to zero the series diverges.

9. $\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n\,(n+1)} \right)$

The first term is a geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{1}{e} \left(\frac{1}{e}\right)^{n-1}$$

Since $r = 1/e$ the series converges.

For the second we use the integral test

$$\int_1^{\infty} \frac{1}{x\,(x+1)}\, dx = \lim_{b\to\infty} \left[\int_1^b \frac{1}{x\,(x+1)}\, dx \right]$$

To solve the integral use partial fractions

$$\lim_{b\to\infty} \left[\int_1^b \frac{1}{x\,(x+1)}\, dx \right] = \lim_{b\to\infty} \left[\int_1^b \frac{1}{x}\, dx - \int_1^b \frac{1}{x+1}\, dx \right] = \lim_{b\to\infty} \left[\ln|x| - \ln|x+1|\right]_1^b$$
$$= \lim_{b\to\infty} \left[\ln\left| \frac{x}{x+1} \right| \right]_1^b$$
$$= \lim_{b\to\infty} \left[\ln\left| \frac{b}{b+1} \right| \right] - \ln\left| \frac{1}{1+1} \right|$$
$$= \ln(1) - \ln\left(\frac{1}{2}\right) = 0 - \ln\left(\frac{1}{2}\right)$$
$$= \ln\left(\frac{1}{2}\right)$$

Since the improper integral converges, the series converges.

Since both series converge, their sum converges.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+7}}$

Use integral test

$$\int_1^{\infty} \frac{1}{\sqrt{x+7}}\,dx = \lim_{b\to\infty}\left[\int_1^b \frac{1}{\sqrt{x+7}}\,dx\right]$$

To solve the integral use the substitution $u = x+7$

$$\lim_{b\to\infty}\left[\int_1^b \frac{1}{\sqrt{x+7}}\,dx\right] = \lim_{b\to\infty}\left[2\sqrt{x+7}\right]_1^b = \lim_{b\to\infty}\left[2\sqrt{b+7}\right] - 2\sqrt{1+7} = \infty$$

Since the improper integral diverges, the series diverges.

11. Find the values of x for which the series converges and find the sum of the series for those values $\displaystyle\sum_{n=1}^{\infty}(x-4)^n$

Rearrange the series to make it a geometric series

$$\sum_{n=1}^{\infty}(x-4)^n = \sum_{n=1}^{\infty}(x-4)(x-4)^{n-1}$$

For $r = x-4$ the series converges if $|x-4| < 1$

$$|x-4| < 1 \qquad \Rightarrow \qquad -1 < x-4 < 1 \qquad \Rightarrow \qquad 3 < x < 5$$

To find the sum

$$\sum_{n=1}^{\infty}(x-4)^n = \frac{a}{1-r} = \frac{x-4}{1-(x-4)} = \frac{x-4}{5-x}$$

The series converges for $3 < x < 5$ and its value is $\dfrac{x-4}{5-x}$.

12. Find the values of x for which the series converges and find the sum of the series for those values $\displaystyle\sum_{n=1}^{\infty}\frac{\cos^n x}{2^n}$

Rearrange the series to make it a geometric series

$$\sum_{n=1}^{\infty}\frac{\cos^n x}{2^n} = \sum_{n=1}^{\infty}\frac{\cos x}{2}\left(\frac{\cos x}{2}\right)^{n-1}$$

For $r = (\cos x)/2$ the series converges if $|(\cos x)/2| < 1$

$$\left|\frac{\cos x}{2}\right| < 1 \qquad \Rightarrow \qquad -1 < \frac{\cos x}{2} < 1 \qquad \Rightarrow \qquad -2 < \cos x < 2$$

Since $-1 < \cos x < 1$, this means that the series converges for all x.

To find the sum

$$\sum_{n=1}^{\infty}\frac{\cos^n x}{2^n} = \frac{a}{1-r} = \frac{\frac{\cos x}{2}}{1-\frac{\cos x}{2}} = \frac{\cos x}{2-\cos x}$$

The series converges for all x and its value is $\dfrac{\cos x}{2-\cos x}$.