1. Find the radius of convergence and interval of convergence of the series

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

- Use Ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{\sqrt{\frac{n+1}{n}}}\right| = \lim_{n\to\infty}\frac{|x|}{\sqrt{1+\frac{1}{n}}} = |x|$$

By the Ratio test, the series converges when $|x| < 1$, so the radius of convergence is $\boxed{R = 1.}$

- Evaluate end points, $x = \pm 1$

When $x = -1$ the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series test.

When $x = 1$ the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges because it is a p-series with $p = \frac{1}{2} < 1$.

The interval of convergence is $\boxed{[-1, 1).}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$

- Use Ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+2}\cdot\frac{n+1}{x^n}\right| = \lim_{n\to\infty}\left(|x|\cdot\frac{n+1}{n+2}\right) = |x|\cdot\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1+\frac{2}{n}} = |x|\cdot 1 = |x|$$

By the Ratio test, the series converges when $|x| < 1$, so the radius of convergence is $\boxed{R = 1.}$

- Evaluate end points, $x = \pm 1$

When $x = -1$ the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+1}$ diverges by the Integral test.

When $x = 1$ the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}$ converges by the Alternating Series test.

The interval of convergence is $\boxed{(-1, 1].}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

- Use Ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = |x|\cdot\lim_{n\to\infty}\frac{1}{n+1} = |x|\cdot 0 = 0 < 1$$

By the Ratio test, the series converges for all x, so the radius of convergence is $\boxed{R = \infty.}$

- Evaluate end points

We don't need to evaluate end points since the radius of convergence is infinity so that the interval of convergence is $\boxed{(-\infty, \infty).}$

(d) $\sum_{n=1}^{\infty} n^n x^n$

- Use Root test (easier)

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{n^n\,|x|^n} = \lim_{n\to\infty} n\,|x| = \infty, \text{ if } x \neq 0$$

So the radius of convergence is $\boxed{R = 0.}$

- Evaluate end points
There are no end points to evaluate since this series only converges when $x = 0$. The interval of convergence is $\boxed{\{0\}.}$

(e) $\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$

- Use Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{5^{n+1}(n+1)^5} \cdot \frac{5^n n^5}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{5}\left(\frac{n}{n+1}\right)^5 = \frac{|x|}{5} \cdot 1 = \frac{|x|}{5}$$

By the Ratio test, the series converges when $\frac{|x|}{5} < 1 \;\Rightarrow\; |x| < 5$, so the radius of convergence is $\boxed{R = 5.}$

- Evaluate end points, $x = \pm 5$
When $x = -5$ the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ converges by the Alternating Series test.
When $x = 5$ the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges because it is a p-series with $p = 5 > 1$.
The interval of convergence is $\boxed{[-5, 5].}$

(f) $\sum_{n=1}^{\infty} \frac{n}{4^n}(x+1)^n$

- Use Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(x+1)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{n\,(x+1)^n}\right| = \frac{|x+1|}{4} \cdot \lim_{n\to\infty} \frac{n+1}{n} = \frac{|x+1|}{4}$$

By the Ratio test, the series converges when $\frac{|x+1|}{4} < 1 \;\Rightarrow\; |x+1| < 4$, so the radius of convergence is $\boxed{R = 4.}$

- Evaluate end points, $|x+1| < 4 \;\Rightarrow\; -4 < x+1 < 4 \;\Rightarrow\; -5 < x < 3$
When $x = -5$ the series $\sum_{n=1}^{\infty} \frac{n\,(-4)^n}{4^n} = \sum_{n=1}^{\infty} (-1)^n\, n$ diverges by the test for Divergence (nth term test).
When $x = 3$ the series $\sum_{n=1}^{\infty} \frac{n\,(4)^n}{4^n} = \sum_{n=1}^{\infty} n$ diverges by the test for Divergence.
The interval of convergence is $\boxed{(-5, 3).}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

- Use Root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-2)^n}{n^n}\right|} = \lim_{n\to\infty} \frac{|x-2|}{n} = 0$$

The series converges for all x, so that $\boxed{R = \infty.}$ and the interval of convergence is $\boxed{(-\infty, \infty).}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{n\,(x-4)^n}{n^3+1}$

- Use Ratio test

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(x-4)^{n+1}}{(n+1)^3+1} \cdot \frac{n^3+1}{n(x-4)^n}\right| &= \lim_{n\to\infty} \left[|x-4| \cdot \frac{n+1}{n} \cdot \frac{n^3+1}{n^3+3n^2+3n+2}\right] \\ &= |x-4| \cdot \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right) \cdot \frac{1+\frac{1}{n^3}}{1+\frac{3}{n}+\frac{3}{n^2}+\frac{2}{n^3}}\right] \\ &= |x-4| \cdot 1 = |x-4|.\end{aligned}$$

By the Ratio test, the series converges when $|x-4| < 1$, so the radius of convergence is $\boxed{R = 1.}$

- Evaluate end points, $|x-1| < 1 \;\Rightarrow\; -1 < x-4 < 1 \;\Rightarrow\; 3 < x < 5$

When $x = 3$ the series $\displaystyle\sum_{n=1}^{\infty} \frac{n\,(-1)^n}{n^3+1}$ converges by the Absolute convergence test.

When $x = 5$ the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ converges by the Comparison test. We compare it to the series $\displaystyle\sum \frac{1}{n^2}$ which is a p-series with $p = 1/2 < 1$ so that is convergent.

The interval of convergence is $\boxed{[3, 5].}$

2. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. What can be said about the convergence or divergence of the following series.

i. Since $\sum c_n\, x^n$ is convergent for $x = -4$ that means it converges for at least $-4 \le x < 4$

ii. Since $\sum c_n\, x^n$ is divergent for $x = 6$ that means it diverges for at least $x \ge 6$ and $x < -6$.

(a) $\displaystyle\sum_{n=0}^{\infty} c_n\, 8^n$ In this case $x = 8$, so that by (ii) above this series diverges

(b) $\displaystyle\sum_{k=0}^{\infty} (-1)^n c_n\, 9^n = \sum_{k=0}^{\infty} c_n\, (-9)^n$ In this case $x = -9$, so that by (ii) above this series diverges

3. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = 2$ and diverges when $x = -3$. What can be said about the convergence or divergence of the following series.

i. Since $\sum c_n\, x^n$ is convergent for $x = 2$ that means it converges for at least $-2 < x \leq 2$

ii. Since $\sum c_n\, x^n$ is divergent for $x = -3$ that means it diverges for at least $x > 3$ and $x \leq -3$.

(c) $\displaystyle\sum_{k=0}^{\infty} \frac{4^n}{5^{n+1}} c_n = \frac{1}{5}\sum_{k=0}^{\infty}\left(\frac{4}{5}\right)^n c_n$ In this case $x = \frac{4}{5}$, so that by (i) above this series converges

(d) $\displaystyle\sum_{n=0}^{\infty} c_n\,(-3)^n$ In this case $x = -3$, so that by (ii) above this series diverges

4. Find a power series representation for the function and determine the interval of convergence

To solve these problems we use the formula for the geometric series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

For this series we know that it converges when $|r| < 1$ or $-1 < r < 1$. The radius of convergence is $R = 1$ and we need to evaluate the end points to determine the interval of convergence.

- When $r = -1$ the series become $\sum_{n=0}^{\infty}(-1)^n$, which is divergent
- When $r = 1$ the series become $\sum_{n=0}^{\infty} 1^n$, which is divergent

Then the Interval of convergence is $I = (-1, 1)$.

(a) $f(x) = \dfrac{1}{1+x}$

Rewrite the function as

$$f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)}$$

So that $r = -x$ and the series representation is

$$f(x) = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n, \text{ the root test gives } |-x| < 1 \text{ or } |x| < 1 \text{ for convergence}$$

And $R = 1$ and $I = (-1, 1)$.

(b) $f(x) = \dfrac{3}{1-x^4}$

Rewrite the function as

$$f(x) = \frac{3}{1-x^4} = 3\left[\frac{1}{1-(x^4)}\right]$$

So that $r = x^4$ and the series representation is

$$f(x) = 3\left[\sum_{n=0}^{\infty}(x^4)^n\right] = \sum_{n=0}^{\infty} 3x^{4n}, \text{ the root test gives } |x^4| < 1 \text{ for convergence } \Rightarrow |x| < 1$$

And $R = 1$ and $I = (-1, 1)$.

(c) $f(x) = \dfrac{x^2}{a^3 - x^3}$

Rewrite the function as

$$f(x) = \frac{x^2}{a^3 - x^3} = \frac{x^2}{a^3} \cdot \frac{1}{1 - \frac{x^3}{a^3}} = \frac{x^2}{a^3} \cdot \sum_{n=0}^{\infty} r^n$$

So that $r = \dfrac{x^3}{a^3}$ and the series representation is

$$f(x) = \frac{x^2}{a^3}\left[\sum_{n=0}^{\infty}\left(\frac{x^3}{a^3}\right)^n\right] = \sum_{n=0}^{\infty}\frac{x^{3n+2}}{a^{3n+3}}.$$

Using the ratio test we get

$$\lim_{n\to\infty}\left|\left[\frac{x^{(3n+3)+2}}{a^{(3n+3)+3}}\right]\left[\frac{a^{3n+3}}{x^{3n+2}}\right]\right| = \lim_{n\to\infty}\left|\frac{a^{3n+3}}{a^{3n+6}} \cdot \frac{x^{3n+5}}{x^{3n+2}}\right| = \lim_{n\to\infty}\left|\frac{x^3}{a^3}\right|, \text{ which gives } \left|\frac{x^3}{a^3}\right| < 1$$
$$\Rightarrow |x^3| < |a^3| \Rightarrow |x| < |a|$$

So that $R = |a|$ and $I = (-|a|, |a|)$.

(d) $f(x) = \ln(5 - x)$

First consider the equality

$$\ln(5 - x) = -\int \frac{1}{5 - x}\,dx$$

We would like to express the integrand as a series, for that we rewrite it as $\dfrac{1}{5 - x} = \dfrac{1}{5} \cdot \dfrac{1}{1 - \frac{x}{5}}$

So that $r = \dfrac{x}{5}$ and the series representation is

$$f(x) = -\frac{1}{5}\int \frac{1}{1 - \frac{x}{5}}\,dx = -\frac{1}{5}\int\left[\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n\right]dx = C - \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=0}^{\infty}\frac{x^{n+1}}{5^{n+1}(n+1)}$$
$$= C - \sum_{n=1}^{\infty}\frac{x^n}{5^n\, n}$$

We find the constant using $f(x) = \ln(5 - x)$; for example for $x = 0$ we have

$$f(0) = \ln(5) = C - \sum_{n=1}^{\infty}\frac{(0)^n}{5^n\, n} = C - 0 = C$$

The the series representation is

$$f(x) = \ln(5) - \sum_{n=1}^{\infty}\frac{x^n}{5^n\, n}.$$

Using the ratio test

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n} \cdot \frac{5^n}{5^{n+1}} \cdot \frac{n+1}{n}\right| = \lim_{n\to\infty}\left|\frac{x}{5} \cdot \frac{n+1}{n}\right| = \left|\frac{x}{5}\right|\lim_{n\to\infty}\frac{n+1}{n} = \left|\frac{x}{5}\right|, \text{ which gives } \left|\frac{x}{5}\right| < 1 \Rightarrow |x| < 5$$

And $R = 5$.

When $x = \pm 5$ the series become $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n}$ which converges by the Alternating Series Test and $I = [-5, 5]$.

(e) $f(x) = \ln(x^2+4)$

First consider the equality
$$f'(x) = \frac{2x}{x^2+4} = \frac{2x}{4}\left[\frac{1}{1-(-\frac{x^2}{4})}\right]$$

We would like to express the term inside the brackets as a series with $r = -\dfrac{x^2}{4}$

$$f'(x) = \frac{2x}{4}\left[\frac{1}{1-(-\frac{x^2}{4})}\right] = \frac{2x}{4}\left[\sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^n\right] = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^{2n+1}}$$

To find $f(x)$ we need to integrate the series

$$f(x) = \int\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^{2n+1}}\,dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{2^{2n+1}(2n+2)} = \ln(4) + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{(n+1)\,2^{2n+2}}$$

Here we use the fact that $f(0) = \ln(4)$, so $C = \ln(4)$.

The series converges when $\left|-\dfrac{x^2}{4}\right| < 1 \Rightarrow x^2 < 4 \Rightarrow |x| < 2$, so $R = 2$.

If $x = \pm 2$ the series becomes $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}$, which converges by the Alternating Series Test and $I = [-2, 2]$.

(f) $f(x) = \dfrac{2x}{(1-x^2)^2}$

First we find the power series representation of the function

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty}x^{2n}, \text{ since } r = x^2$$

Next we multiply this series by itself to find the quadratic form $\dfrac{1}{(1-x^2)^2}$,

$$\begin{aligned}\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^2}\right) &= (1+x^2+x^4+x^6+\cdots)\,(1+x^2+x^4+x^6+\cdots)\\ &= 1+x^2+x^4+x^6+x^2+x^4+x^6+x^8+x^4+x^6+x^8+x^{10}+x^6+x^8+x^{10}+x^{12}+\cdots\\ &= 1+2x^2+3x^4+4x^6+5x^8+\cdots = \sum_{n=0}^{\infty}(n+1)x^{2n}\end{aligned}$$

To find $f(x)$ we need to multiply the series by $2x$

$$f(x) = \sum_{n=0}^{\infty}2\,(n+1)\,x^{2n+1}.$$

Using the ratio test

$$\lim_{n\to\infty}\left|\frac{x^{2n+3}}{x^{2n+1}}\cdot\frac{n+2}{n+1}\right| = |x^2|\lim_{n\to\infty}\frac{n+2}{n+1} = |x^2|.$$

The series converges when $|x^2| < 1$, so $R = 1$.

If $x = \pm 1$ the series becomes $\displaystyle\sum_{n=0}^{\infty}\pm 2(n+1)$, which diverge by nth-test and $I = (-1, 1)$.

(g) $f(x) = \tan^{-1}(x^2)$

First consider the equality $$f'(x) = \frac{2x}{x^4+1} = 2x\left[\frac{1}{1-(-x^4)}\right]$$

We would like to express the term inside the brackets as a series with $r = -x^4$

$$f'(x) = 2x\left[\frac{1}{1-(-x^4)}\right] = 2x\left[\sum_{n=0}^{\infty}\left(-x^4\right)^n\right] = 2x\left[\sum_{n=0}^{\infty}(-1)^n\, x^{4n}\right] = \sum_{n=0}^{\infty}(-2)^n\, x^{4n+1}$$

To find $f(x)$ we need to integrate the series

$$f(x) = \int\sum_{n=0}^{\infty}(-2)^n\, x^{4n+1}\, dx = C + \sum_{n=0}^{\infty}(-2)^n\,\frac{x^{4n+2}}{4n+2} = \sum_{n=0}^{\infty}(-2)^n\,\frac{x^{4n+2}}{4n+2} = \sum_{n=0}^{\infty}(-1)^n\,\frac{x^{4n+2}}{2n+1}$$

Here we use the fact that $f(0) = \tan^{-1}(0) = 0$, so $C = 0$.

Using the ration test

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{x^{4n+6}}{x^{4n+2}}\cdot\frac{2n+3}{2n+1}\right| = \left|(-1)x^4\right|\lim_{n\to\infty}\frac{2n+3}{2n+1} = |x^4|.$$

The series converges when $\left|x^4\right| < 1$, so $R = 1$.

If $x = \pm 1$ the series becomes $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$, which converges by the Alternating Series Test and $I = [-1, 1]$.

5. Evaluate the integral as a power series. What is the radius of convergence?

(a) $\displaystyle\int \frac{t}{1-t^8}\,dt$

First we express the integrand as a power series:

$$\frac{t}{1-t^8} = t \cdot \frac{1}{1-(t^8)} = t \cdot \sum_{n=0}^{\infty} (t^8)^n = t \cdot \sum_{n=0}^{\infty} t^{8n} = \sum_{n=0}^{\infty} t^{8n+1}$$

The series for $\dfrac{1}{1-t^8}$ converges when $|t^8| < 1$ or $|t| < 1$, so $R = 1$.

Next we evaluate the integral

$$\int \frac{t}{1-t^8}\,dt = \int \left[\sum_{n=0}^{\infty} t^{8n+1}\right] dt = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$$

(b) $\displaystyle\int \frac{\ln(1-t)}{t}\,dt$

First we express the integrand as a power series, for that we only consider the natural log:

$$\ln(1-t) = -\int \frac{1}{1-t}\,dt = -\int \left[\sum_{n=0}^{\infty} t^n\right] dt = C - \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} = -\sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1}$$

The series for converges when $|t| < 1$ or $|t| < 1$, so $R = 1$.

Next we evaluate the integral

$$\int \frac{\ln(1-t)}{t}\,dt = \int \left[\frac{1}{t} \cdot \left(-\sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1}\right)\right] dt = -\int \left[\sum_{n=0}^{\infty} \frac{t^n}{n+1}\right] dt = C + \sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)^2}$$

(c) $\displaystyle\int \frac{x^2}{1+x^4}\,dx$

First we express the integrand as a power series:

$$\frac{x^2}{1+x^4} = x^2 \cdot \frac{1}{1-(-x^4)} = x^2 \cdot \sum_{n=0}^{\infty} \left(-x^4\right)^n = x^2 \cdot \sum_{n=0}^{\infty} (-1)^n x^{4n} = \sum_{n=0}^{\infty} (-1)^n x^{4n+2}$$

The series for converges when $|-x^4| = |x^4| < 1$ or $|x| < 1$, so $R = 1$.

Next we evaluate the integral

$$\int \frac{x^2}{1+x^4}\,dx = \int \left[\sum_{n=0}^{\infty} (-1)^n x^{4n+2}\right] dx = C + (-1)^n \frac{x^{4n+3}}{4n+3}$$

6. Use partial fractions to find the power series of each of the following functions.

(a) $\dfrac{4}{(x-3)(x+1)}$

$$\frac{4}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \qquad \Rightarrow \qquad 4 = A(x+1) + B(x-3)$$

which gives $A = 1$ and $B = -1$.

Next we find the two power series:

$$\frac{1}{x-3} = -\frac{1}{3}\left[\frac{1}{1-\left(\frac{x}{3}\right)}\right] = -\frac{1}{3}\left[\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n\right] = -\sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}$$

$$-\frac{1}{x+1} = -\frac{1}{1-(-x)} = -\sum_{n=0}^{\infty}(-x)^n = -\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-1)^{n+1}x^n$$

Then the power series representation of the function is

$$f(x) = \frac{4}{(x-3)(x+1)} = -\frac{1}{x+1} + \frac{1}{x-3} = \sum_{n=0}^{\infty}(-1)^{n+1}x^n - \sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}} = \sum_{n=0}^{\infty}\left[(-1)^{n+1} - \frac{1}{3^{n+1}}\right]x^n$$

(b) $\dfrac{5}{(x^2+4)(x^2-1)}$

$$\frac{5}{(x^2+4)(x^2-1)} = \frac{5}{(x^2+4)(x-1)(x+1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-1} + \frac{D}{x+1}$$

$$5 = (Ax+B)(x^2-1) + C\,(x^2+4)(x+1) + D\,(x^2+4)(x-1)$$

which gives $A = 0$, $B = -1$, $C = 1/2$, and $D = -1/2$

The individual power series are given by

$$-\frac{1}{4+x^2} = -\frac{1}{4}\left[\frac{1}{1-\left(-\frac{x^2}{4}\right)}\right] = -\frac{1}{4}\left[\sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^n\right] = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{4^{n+1}}$$

$$\frac{1}{2(x-1)} = -\frac{1}{2}\left[\frac{1}{1-x}\right] = -\frac{1}{2}\left[\sum_{n=0}^{\infty}x^n\right] = -\frac{1}{2}\sum_{n=0}^{\infty}x^n$$

$$-\frac{1}{2(x+1)} = -\frac{1}{2}\left[\frac{1}{1-(-x)}\right] = -\frac{1}{2}\left[\sum_{n=0}^{\infty}(-x)^n\right] = -\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^n$$

Then the power series representation of the function is

$$f(x) = \frac{5}{(x^2+4)(x^2-1)} = -\frac{1}{4+x^2} + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{4^{n+1}} - \frac{1}{2}\sum_{n=0}^{\infty}x^n - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{4^{n+1}} - 1\right]x^{2n}$$

(c) $\dfrac{3}{(x+2)(x-1)}$

$$\begin{aligned}\frac{3}{(x+2)(x-1)} &= \frac{A}{x+2} + \frac{B}{x-1} \\ 3 &= A(x-1) + B(x+2)\end{aligned}$$

which gives $A = -1$ and $B = 1$.

The individual power series are given by

$$-\frac{1}{x+2} = -\frac{1}{2}\left[\frac{1}{1-\left(-\frac{x}{2}\right)}\right] = -\frac{1}{2}\left[\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n\right] = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{2^{n+1}}$$

$$\frac{1}{x-1} = -\frac{1}{1-x} = -\sum_{n=0}^{\infty} x^n$$

Then the power series representation of the function is

$$\begin{aligned} f(x) &= \frac{3}{(x+2)(x-1)} = -\frac{1}{x+2} + \frac{1}{x-1} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{2^{n+1}} - \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty}\left[\left(-\frac{1}{2}\right)^{n+1} - 1\right] x^n \end{aligned}$$