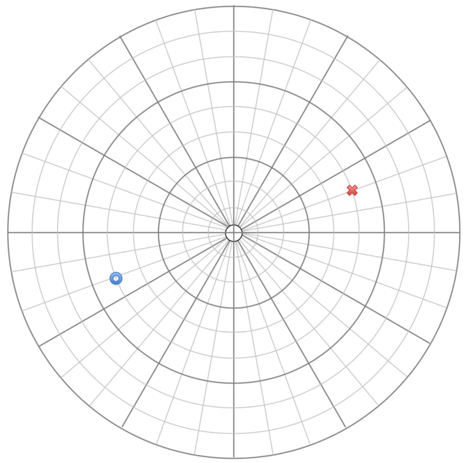
1. Which of the following polar coordinate pairs represent the same point as the point with polar coordinates (r, θ)? Assume $r \neq 0$

(a) $(-r, \theta)$ (b) $(r, \theta + \pi)$ (c) $(-r, \theta + 2\pi)$ (d) $(r, \theta + 3\pi)$ (e) $(-r, \theta + 3\pi)$

The corresponding point is **(e)** $\mathbf{(-r, \boldsymbol{\theta} + 3\boldsymbol{\pi})}$. *All other points represent the point marked as a circle in the figure below.*



2. Which of the following polar coordinate pairs does NOT represent the point with rectangular coordinates $(-2, -2)$?

The answer is (d).

(a) $(-2\sqrt{2}, \pi/4)$

$$\begin{aligned} x &= r\cos\theta = -2\sqrt{2}\cdot\cos(\pi/4) = -2\sqrt{2}\cdot\frac{\sqrt{2}}{2} = -2 \\ y &= r\sin\theta = -2\sqrt{2}\cdot\sin(\pi/4) = -2\sqrt{2}\cdot\frac{\sqrt{2}}{2} = -2 \end{aligned}$$

(b) $(-2\sqrt{2}, -7\pi/4)$

$$\begin{aligned} x &= r\cos\theta = -2\sqrt{2}\cdot\cos(-7\pi/4) = -2\sqrt{2}\cdot\cos(7\pi/4) = -2\sqrt{2}\cdot\frac{\sqrt{2}}{2} = -2 \\ y &= r\sin\theta = -2\sqrt{2}\cdot\sin(-7\pi/4) = -2\sqrt{2}\cdot(-\sin(7\pi/4)) = -2\sqrt{2}\cdot-\left(-\frac{\sqrt{2}}{2}\right) = -2 \end{aligned}$$

(c) $(2\sqrt{2}, -3\pi/4)$

$$\begin{aligned} x &= r\cos\theta = 2\sqrt{2}\cdot\cos(-3\pi/4) = 2\sqrt{2}\cdot\cos(3\pi/4) = 2\sqrt{2}\cdot-\frac{\sqrt{2}}{2} = -2 \\ y &= r\sin\theta = 2\sqrt{2}\cdot\sin(-3\pi/4) = 2\sqrt{2}\cdot(-\sin(3\pi/4)) = 2\sqrt{2}\cdot-\frac{\sqrt{2}}{2} = -2 \end{aligned}$$

(d) $(-2\sqrt{2}, 3\pi/4)$

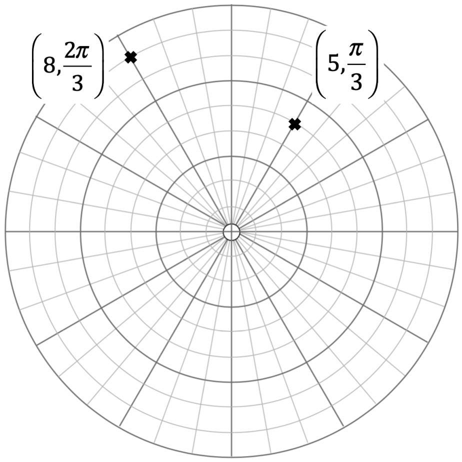
$$\begin{aligned} x &= r\cos\theta = -2\sqrt{2}\cdot\cos(3\pi/4) = -2\sqrt{2}\cdot-\frac{\sqrt{2}}{2} = 2 \\ y &= r\sin\theta = -2\sqrt{2}\cdot\sin(3\pi/4) = -2\sqrt{2}\cdot\frac{\sqrt{2}}{2} = -2 \end{aligned}$$

(e) $(2\sqrt{2}, 5\pi/4)$

$$\begin{aligned} x &= r\cos\theta = 2\sqrt{2}\cdot\cos(5\pi/4) = 2\sqrt{2}\cdot-\frac{\sqrt{2}}{2} = -2 \\ y &= r\sin\theta = 2\sqrt{2}\cdot\sin(5\pi/4) = 2\sqrt{2}\cdot-\frac{\sqrt{2}}{2} = -2 \end{aligned}$$

3. A radar detects two airplanes at the same altitude. Their polar coordinates are (in miles) $(8, 2\pi/3)$ and $(5, \pi/3)$. How far apart are the airplanes?

The distance is 7 miles.

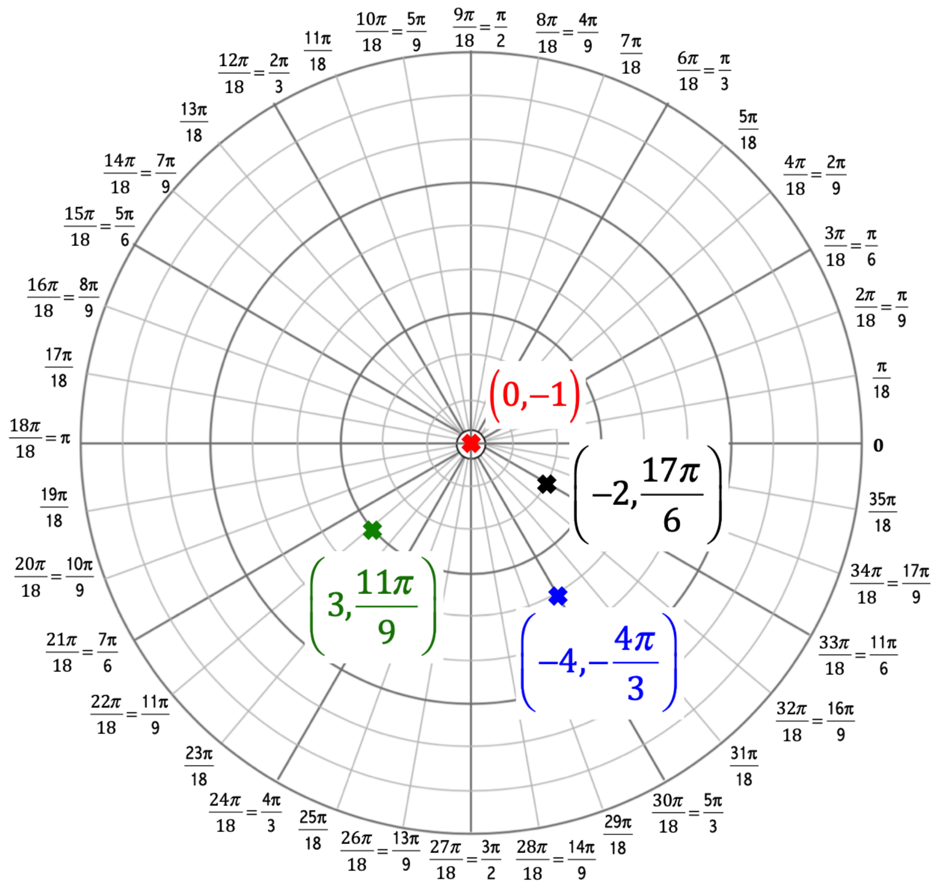


$$\begin{aligned} x_1 &= 8 \cdot \cos(2\pi/3) = 8 \cdot -\frac{1}{2} = -4 \\ y_1 &= 8 \cdot \sin(2\pi/3) = 8 \cdot \frac{\sqrt{3}}{2} = 4\sqrt{3} \end{aligned}$$

$$\begin{aligned} x_2 &= 5 \cdot \cos(\pi/3) = 5 \cdot \frac{1}{2} = \frac{5}{2} \\ y_2 &= 5 \cdot \sin(\pi/3) = 5 \cdot \frac{\sqrt{3}}{2} = \frac{5}{2}\sqrt{3} \end{aligned}$$

$$\begin{aligned} d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{\left(-4 - \frac{5}{2}\right)^2 + \left(4\sqrt{3} - \frac{5}{2}\sqrt{3}\right)^2} = \sqrt{\left(-\frac{13}{2}\right)^2 + \left(\frac{3}{2}\sqrt{3}\right)^2} \\ &= \sqrt{\frac{169}{4} + \frac{27}{4}} = \sqrt{\frac{196}{4}} = \frac{14}{2} = 7. \end{aligned}$$

4. Plot the points for the given polar coordinates



5. Find the rectangular coordinates for the point whose polar coordinates are given

(a) $(7, 7\pi) = (-7, 0)$

$$\begin{aligned} x &= r\cos\theta = 7 \cdot \cos(7\pi) = 7 \cdot -1 = -7 \\ y &= r\sin\theta = 7 \cdot \sin(7\pi) = 7 \cdot 0 = 0 \end{aligned}$$

(b) $\left(7, \frac{\pi}{6}\right) = \left(\frac{7\sqrt{3}}{2}, \frac{7}{2}\right)$

$$\begin{aligned} x &= r\cos\theta = 7 \cdot \cos(\pi/6) = 7 \cdot \frac{\sqrt{3}}{2} = \frac{7}{2}\sqrt{3} \\ y &= r\sin\theta = 7 \cdot \sin(\pi/6) = 7 \cdot \frac{1}{2} = \frac{7}{2} \end{aligned}$$

(c) $\left(\sqrt{2}, -\frac{3\pi}{4}\right) = (-1, 1)$

$$\begin{aligned} x &= r\cos\theta = \sqrt{2} \cdot \cos(3\pi/4) = \sqrt{2} \cdot -\frac{\sqrt{2}}{2} = -1 \\ y &= r\sin\theta = \sqrt{2} \cdot \sin(3\pi/4) = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 1 \end{aligned}$$

(d) $\left(-\sqrt{3}, \frac{5\pi}{3}\right) = \left(-\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$

$$\begin{aligned} x &= r\cos\theta = -\sqrt{3} \cdot \cos(5\pi/3) = -\sqrt{3} \cdot \frac{1}{2} = -\frac{\sqrt{3}}{2} \\ y &= r\sin\theta = -\sqrt{3} \cdot \sin(5\pi/3) = -\sqrt{3} \cdot -\frac{\sqrt{3}}{2} = \frac{3}{2} \end{aligned}$$

6. Convert the rectangular coordinates to polar coordinates with $r > 0$ and $0 \leq \theta \leq 2\pi$.

(a) $\left(-\sqrt{3}, 0\right) = \left(\sqrt{3}, 0\right) = \left(\sqrt{3}, 2\pi\right)$

$$r = \sqrt{x^2+y^2} = \sqrt{3+0} = \sqrt{3}$$

$$\sin\theta = \frac{y}{r} = \frac{0}{\sqrt{3}} \Rightarrow \theta = 0, \pi \text{ or } 2\pi \qquad \cos\theta = \frac{x}{r} = \frac{-\sqrt{3}}{\sqrt{3}} = -1 \Rightarrow \theta = \pi$$

(b) $(-6, 6) = \left(6\sqrt{2}, \frac{3\pi}{4}\right)$

$$r = \sqrt{x^2+y^2} = \sqrt{36+36} = 6\sqrt{2}$$

$$\sin\theta = \frac{y}{r} = \frac{6}{6\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \qquad \cos\theta = \frac{x}{r} = \frac{-6}{6\sqrt{2}} = -\frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{3\pi}{4} \text{ or } \frac{5\pi}{4}$$

(c) $\left(\sqrt{8}, -\sqrt{8}\right) = \left(4, \frac{7\pi}{4}\right)$

$$r = \sqrt{x^2+y^2} = \sqrt{8+8} = \sqrt{16} = 4$$

$$\sin\theta = \frac{y}{r} = \frac{-\sqrt{8}}{4} = -\frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{5\pi}{4} \text{ or } \frac{7\pi}{4} \qquad \cos\theta = \frac{x}{r} = \frac{\sqrt{8}}{4} = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{7\pi}{4}$$

(d) $\left(\sqrt{6}, \sqrt{2}\right) = \left(2\sqrt{2}, \frac{\pi}{6}\right)$

$$r = \sqrt{x^2+y^2} = \sqrt{6+2} = \sqrt{8} = 2\sqrt{2}$$

$$\sin\theta = \frac{y}{r} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \qquad \cos\theta = \frac{x}{r} = \frac{\sqrt{2\times 3}}{2\sqrt{2}} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{11\pi}{6}$$

7. Convert the polar equation to rectangular form

(a) $r^2 = \sin(2\theta)$

$$\begin{aligned} r^2 &= \sin(2\theta) = 2\sin\theta\cos\theta \\ r^2 \cdot r^2 &= 2\,(r\sin\theta)\,(r\cos\theta) \\ (x^2+y^2)^2 &= 2xy \end{aligned}$$

(b) $r = 5$

Circle of radius 5, center at the origin: $x^2 + y^2 = 25$

(c) $\theta = \dfrac{4\pi}{3}$

Line that intersects the origin with slope $\tan(\frac{4\pi}{3}) = \sqrt{3}$*:* $y = \sqrt{3}\,x$

(d) $\csc\theta = 2$

$$\begin{aligned} \csc\theta &= 2 \\ \frac{1}{\sin\theta} &= 2 \\ 1 &= 2\sin\theta \\ r &= 2r\sin\theta \\ \sqrt{x^2+y^2} &= 2y \end{aligned}$$

(e) $r = \dfrac{4}{1+2\sin\theta}$

$$\begin{aligned} r &= \frac{4}{1+2\sin\theta} \\ r + 2r\sin\theta &= 4 \\ \sqrt{x^2+y^2} + 2y &= 4 \end{aligned}$$

(f) $r = 3\cos\theta$

$$\begin{aligned} r &= 3\cos\theta \\ r^2 &= 3r\cos\theta \\ \sqrt{x^2+y^2} &= 3x \end{aligned}$$

8. Convert the rectangular equation to polar form.

(a) $x^2 - y^2 = 1$

$$\begin{aligned} x^2 - y^2 &= 1 \\ r^2\cos^2\theta - r^2\sin^2\theta = r^2\left(\cos^2\theta - \sin^2\theta\right) &= 1 \\ r^2\cos(2\theta) &= 1 \end{aligned}$$

(b) $y = x$

$$\begin{aligned} y &= x \\ r\sin\theta &= r\cos\theta \\ \tan\theta &= 1 \quad \text{or} \quad \theta = \pi/4, 5\pi/4 \quad \text{and} \quad r \neq 0 \end{aligned}$$

(c) $x = 8$

$$\begin{aligned} x &= 8 \\ r\cos\theta &= 8 \end{aligned}$$

(d) $y = 2x^2$

$$\begin{aligned} y &= 2x^2 \\ r\sin\theta &= 2r^2\cos^2\theta \\ 2r\cos\theta &= \tan\theta \quad \text{and} \quad r \neq 0 \end{aligned}$$

9. Find the points at which the curve given by $r = \sin\theta + \cos\theta$ has a vertical or horizontal tangent line.

Since $x = r\cos\theta$ and $y = r\sin\theta$, we can rewrite these equations as

$$x = r\cos\theta = (\sin\theta + \cos\theta)\cos\theta = \sin\theta\,\cos\theta + \cos^2\theta \quad \text{and} \quad \frac{dx}{d\theta} = \cos^2\theta - \sin^2\theta - 2\cos\theta\,\sin\theta$$

$$y = r\sin\theta = (\sin\theta + \cos\theta)\sin\theta = \sin\theta\,\cos\theta + \sin^2\theta \quad \text{and} \quad \frac{dy}{d\theta} = \cos^2\theta - \sin^2\theta + 2\cos\theta\,\sin\theta$$

And the slope of the tangent line is given by

$$\frac{dy}{dt} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos^2\theta - \sin^2\theta + 2\cos\theta\,\sin\theta}{\cos^2\theta - \sin^2\theta - 2\cos\theta\,\sin\theta}$$

*For a horizontal tangent line, we need the slope to be zero, that means the numerator (**but not the denominator**) should be zero.*

$$\begin{aligned} \cos^2\theta - \sin^2\theta + 2\cos\theta\,\sin\theta &= 0 \\ \cos(2\theta) + \sin(2\theta) &= 0 \\ \cos(2\theta) &= -\sin(2\theta) \end{aligned}$$

In the interval $[0, 2\pi)$ this is true for $2\theta = 3\pi/4$ and $2\theta = 7\pi/4$. We still need to check that the denominator is not zero for these values:

$$\begin{aligned} \cos^2\theta - \sin^2\theta - 2\cos\theta\,\sin\theta &= \cos(2\theta) - \sin(2\theta) = \cos(2\theta) - (-\cos(2\theta)) = \cos(2\theta) + \cos(2\theta) \\ &= 2\cos(2\theta) \neq 0 \text{ for } 2\theta = 3\pi/4 \text{ or } 2\theta = 7\pi/4 \end{aligned}$$

- *For $2\theta = 3\pi/4$, we have $\theta = 3\pi/8$ and $r = \sin(3\pi/8) + \cos(3\pi/8)$.*
- *Similarly for $2\theta = 7\pi/4$, we have $\theta = 7\pi/8$ and $r = \sin(7\pi/8) + \cos(7\pi/8)$.*

The plot has horizontal tangents at

$$\left(\sin(3\pi/8) + \cos(3\pi/8), \frac{3\pi}{8}\right) \quad \text{and} \quad \left(\sin(7\pi/8) + \cos(7\pi/8), \frac{7\pi}{8}\right)$$

*For a vertical tangent line, we need the slope to be undefined, that means the denominator (**but not the numerator**) should be zero.*

$$\begin{aligned} \cos^2\theta - \sin^2\theta - 2\cos\theta\,\sin\theta &= 0 \\ \cos(2\theta) - \sin(2\theta) &= 0 \\ \cos(2\theta) &= \sin(2\theta) \end{aligned}$$

In the interval $[0, 2\pi)$ this is true for $2\theta = \pi/4$ and $2\theta = 5\pi/4$. We still need to check that the denominator is not zero for these values:

$$\begin{aligned} \cos^2\theta - \sin^2\theta + 2\cos\theta\,\sin\theta &= \cos(2\theta) + \sin(2\theta) = \cos(2\theta) + (\cos(2\theta)) = \cos(2\theta) + \cos(2\theta) \\ &= 2\cos(2\theta) \neq 0 \text{ for } 2\theta = \pi/4 \text{ or } 2\theta = 5\pi/4 \end{aligned}$$

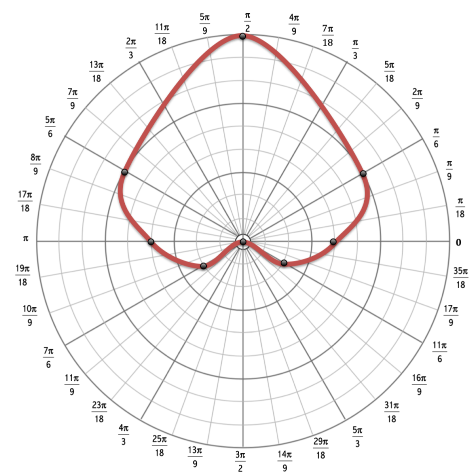
The plot has vertical tangents at

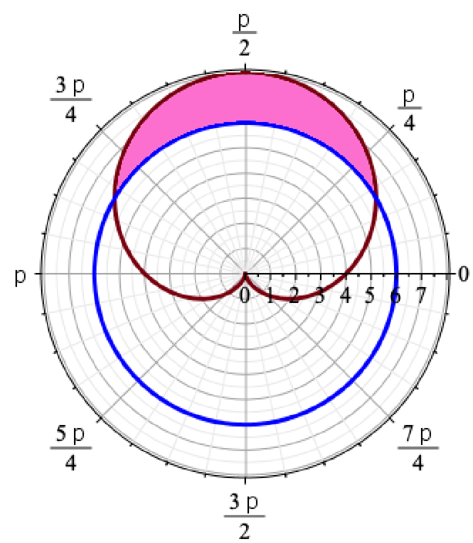
$$\left(\sin(\pi/8) + \cos(\pi/8), \frac{\pi}{8}\right) \quad \text{and} \quad \left(\sin(5\pi/8) + \cos(5\pi/8), \frac{5\pi}{8}\right)$$

10. Sketch the region that lies inside the curve $r = 4 + 4\sin\theta$ and outside the curve $r = 6$.

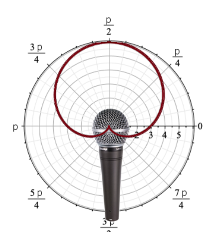
The second curve is a circle of radius 6 centered at the origin, and we can sketch the first curve as,

θ	$r = 4 + 4\sin\theta$
0	$4+0=4$
$\pi/6$	$4+2=6$
$\pi/2$	$4+4=8$
$5\pi/6$	$4+2=6$
π	$4+0=4$
$7\pi/6$	$4-2=2$
$3\pi/2$	$4-4=0$
$11\pi/6$	$4-2=2$





11. A microphone's directionality or polar pattern indicates how sensitive it is to sounds arriving at different angles about its central axis. The polar pattern illustrated below represents the locus of points that produce the same signal level output in the microphone if a given sound pressure level (SPL) is generated from that point.



Which polar equation best describes the microphone directionality? ***(b)***

(a) $r = 6$: *Circle of radius 6, centered at (0,0)*

(b) $r = 3 + 3\sin\theta$: *Cardioid - see previous problem*

(c) $r = 6\cos\theta$: *Circle of radius 3, centered at (3,0)*

$$\begin{aligned} r = 6\cos\theta \;\Rightarrow\; r^2 = 6r\cos\theta \;\Rightarrow\; x^2 + y^2 &= 6x \\ x^2 + y^2 - 6x &= 0 \\ (x^2 - 6x + 9) + y^2 &= 9 \\ (x-3)^2 + y^2 &= 3^2 \end{aligned}$$