Evaluate the following integrals

1. $\displaystyle\int \frac{-2x^2+19x-13}{(x-5)(x-1)^2}\,dx$

$$\frac{-2x^2+19x-13}{(x-5)(x-1)^2} = \frac{A}{x-5}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

Multiply both sides by the denominator of the right hand side:

$$\begin{aligned}\frac{-2x^2+19x-13}{(x-5)(x-1)^2}\cdot\left[(x-5)(x-1)^2\right] &= \frac{A}{x-5}\cdot\left[(x-5)(x-1)^2\right]+\frac{B}{x-1}\cdot\left[(x-5)(x-1)^2\right]\\ &\quad+\frac{C}{(x-1)^2}\cdot\left[(x-5)(x-1)^2\right]\end{aligned}$$

Canceling terms

$$\begin{aligned}\frac{-2x^2+19x-13}{\cancel{(x-5)(x-1)^2}}\cdot\left[\cancel{(x-5)(x-1)^2}\right] &= \frac{A}{\cancel{x-5}}\cdot\left[\cancel{(x-5)}(x-1)^2\right]+\frac{B}{\cancel{x-1}}\cdot\left[(x-5)(x-1)\cancel{(x-1)}\right]\\ &\quad+\frac{C}{\cancel{(x-1)^2}}\cdot\left[(x-5)\cancel{(x-1)^2}\right]\end{aligned}$$

$$\begin{aligned}-2x^2+19x-13 &= A(x-1)^2+B(x-5)(x-1)+C(x-5)\\ &= Ax^2-2Ax+A+Bx^2-6Bx+5B+Cx-5C\\ &= (A+B)\,x^2+(-2A-6B+C)\,x+(-A-5B-5C)\end{aligned}$$

- Coefficient of x^2: $\quad -2 = A + B$
- Coefficient of x^1: $\quad 19 = -2A - 6B + C$
- Coefficient of x^0: $\quad -13 = A + 5B - 5C$

From the first equation $\quad B = -2 - A$

From the second equation $\quad \begin{aligned} C &= 19+2A+6B = 19+2A+6\,(-2-A) = 19+2A-12-6A\\ &= 7-4A\end{aligned}$

From the third equation $\quad \begin{aligned} -13 &= A+5B-5C = A+5(-2-A)-5(7-4A)\\ &= A-10-5A-35+20A = -45+16A\\ -13+45 &= 16A\\ 32 &= 16A\end{aligned}$

$$\begin{aligned} A &= 2\\ B=-2-A=-2-2 &= -4\\ C=7-4A=7-4\cdot 2 &= -1\end{aligned}$$

And the integral becomes

$$\begin{aligned}\int\frac{-2x^2+19x-13}{(x-5)(x-1)^2}\,dx &= \int\frac{2}{x-5}\,dx-\int\frac{4}{x-1}\,dx-\int\frac{1}{(x-1)^2}\,dx\\ &= \boxed{2\ln|x-5|-4\ln|x-1|+\frac{1}{x-1}+C.}\end{aligned}$$

2. $\displaystyle\int \frac{20x+9}{25x^2+20x+4}\,dx$

First, factor the denominator as $25x^2+20x+4=(5x+2)^2$, so that

$$\frac{20x+9}{25x^2+20x+4}=\frac{20x+9}{(5x+2)^2}=\frac{A}{5x+2}+\frac{B}{(5x+2)^2}$$

Multiply both sides by the denominator of the right hand side:

$$\frac{20x+9}{(5x+2)^2}\cdot\left[(5x+2)^2\right]=\frac{A}{5x+2}\cdot\left[(5x+2)^2\right]+\frac{B}{(5x+2)^2}\cdot\left[(5x+2)^2\right]$$

Canceling terms

$$\begin{aligned}\frac{20x+9}{\cancel{(5x+2)^2}}\cdot\left[\cancel{(5x+2)^2}\right] &= \frac{A}{\cancel{5x+2}}\cdot\left[\cancel{(5x+2)}(5x+2)\right]+\frac{B}{\cancel{(5x+2)^2}}\cdot\left[\cancel{(5x+2)^2}\right]\\ 20x+9 &= 5Ax+2A+B\end{aligned}$$

- From the coefficients of x^1 we get $20=5A$, which gives us $A=4$.
- From the coefficients of x^0 we get $9=2A+B=8+B$. which gives us $B=1$.

And the integral becomes

$$\begin{aligned}\int\frac{20x+9}{25x^2+20x+4}\,dx &= \int\frac{4}{5x+2}\,dx+\int\frac{1}{(5x+2)^2}\,dx\\ &= \boxed{\frac{4}{5}\ln|5x+2|-\frac{1}{5(5x+2)}+C.}\end{aligned}$$

3. $\displaystyle\int \frac{-5x+4}{x^2-x}\,dx$

First, factor the denominator as $x^2-x=x\,(x-1)$, so that $\displaystyle\frac{-5x+4}{x^2-x}=\frac{-5x+4}{x\,(x-1)}=\frac{A}{x}+\frac{B}{x-1}$

Multiply both sides by the denominator of the right hand side:

$$\frac{-5x+4}{x\,(x-1)}\cdot[x(x-1)]=\frac{A}{x}\cdot[x(x-1)]+\frac{B}{x-1}\cdot[x(x-1)]$$

Canceling terms

$$\begin{aligned}\frac{-5x+4}{\cancel{x\,(x-1)}}\cdot[\cancel{x(x-1)}] &= \frac{A}{\cancel{x}}\cdot[\cancel{x}(x-1)]+\frac{B}{\cancel{x-1}}\cdot[x\cancel{(x-1)}]\\ -5x+4 &= Ax-A+Bx\end{aligned}$$

- From the coefficients of x^1 we get $-5=A+B$, which gives us $B=-5-A$.
- From the coefficients of x^0 we get $4=-A$, which gives us $A=-4$ and $B=-1$.

And the integral becomes

$$\begin{aligned}\int\frac{-5x+4}{x^2-x}\,dx &= -\int\frac{4}{x}\,dx-\int\frac{1}{x-1}\,dx\\ &= \boxed{-4\ln|x|-\ln|x-1|+C.}\end{aligned}$$

4. $\displaystyle\int \frac{-2x^2+4x+14}{x^2-6x+5}\,dx$

Since this is an improper fraction, we first need to perform long division

$$\begin{array}{r} -2 \\ x^2-6x+5\overline{\big)\,-2x^2+4x+14} \\ 2x^2-12x+10 \\ \hline -8x+24 \end{array}$$

$$\frac{-2x^2+4x+14}{x^2-6x+5} = -2 + \frac{-8x+24}{x^2-6x+5}$$

To find the partial fractions, we need to factor the denominator as $x^2-6x+5=(x-1)(x-5)$, so that

$$\frac{-8x+24}{x^2-6x+5} = \frac{-8x+24}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5}$$

Multiply both sides by the denominator of the right hand side:

$$\frac{-8x+24}{x^2-6x+5}\cdot[(x-1)(x-5)] = \frac{A}{x-1}\cdot[(x-1)(x-5)] + \frac{B}{x-5}\cdot[(x-1)(x-5)]$$

Canceling terms
$$\frac{-8x+24}{\cancel{(x-1)(x-5)}}\cdot[\cancel{(x-1)(x-5)}] = \frac{A}{\cancel{x-1}}\cdot[\cancel{(x-1)}(x-5)] + \frac{B}{\cancel{x-5}}\cdot[(x-1)\cancel{(x-5)}]$$
$$-8x+24 = Ax-5A+Bx-B$$

- From the coefficients of x^1 we get $-8=A+B$, which gives us $B=-8-A$.
- From the coefficients of x^0 we get $24=-5A-B=-5A+8+A$, which gives us $A=-4$ and $B=-4$.

And the integral becomes

$$\int \frac{-2x^2+4x+14}{x^2-6x+5}\,dx = \int\left(-2+\frac{-8x+24}{x^2-6x+5}\right)dx = \int\left(-2-\frac{4}{x-1}-\frac{4}{x-5}\right)dx$$
$$= \boxed{-2x-4\ln|x-1|-4\ln|x-5|+C.}$$

5. $\displaystyle\int \frac{-7x-15}{x^2+6x+9}\,dx$

First, factor the denominator as $x^2+6x+9=(x+3)^2$, so that $\dfrac{-7x-15}{x^2+6x+9} = \dfrac{-7x-15}{(x+3)^2} = \dfrac{A}{x+3} + \dfrac{B}{(x+3)^2}$

Multiply both sides by the denominator of the right hand side:

$$\frac{-7x-15}{(x+3)^2}\cdot[(x+3)^2] = \frac{A}{x+3}\cdot[(x+3)^2] + \frac{B}{(x+3)^2}\cdot[(x+3)^2]$$

Canceling terms
$$\frac{-7x-15}{\cancel{(x+3)^2}}\cdot[\cancel{(x+3)^2}] = \frac{A}{\cancel{x+3}}\cdot[\cancel{(x+3)}(x+3)] + \frac{B}{\cancel{(x+3)^2}}\cdot[\cancel{(x+3)^2}]$$
$$-7x-15 = Ax+3A+B$$

- From the coefficients of x^1 we get $-7=A$.
- From the coefficients of x^0 we get $-15=3A+B$, which gives us $B=-15-3A=6$.

And the integral becomes

$$\int \frac{-7x-15}{x^2+6x+9}\,dx = \int\left(\frac{-7}{x+3}+\frac{6}{(x+3)^2}\right)dx = \boxed{-7\ln|x+3|-\frac{6}{x+3}+C.}$$

6. $\displaystyle\int \frac{-6x^2+3x+5}{x^3-x}\,dx$

First, factor the denominator as $x^3 - x = x(x-1)(x+1)$,

so that $\dfrac{-6x^2+3x+5}{x\,(x^2-1)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1}$

Multiply both sides by the denominator of the right hand side:

$$\frac{-6x^2+3x+5}{x\,(x-1)(x+1)}\cdot[x\,(x-1)^2] \quad = \quad \frac{A}{x}\cdot[x\,(x-1)(x+1)] + \frac{B}{x-1}\cdot[x\,(x-1)(x+1)] + \frac{C}{x+1}\cdot[x\,(x-1)(x+1)]$$

Canceling terms

$$\begin{aligned}
\frac{-6x^2+3x+5}{\cancel{x\,(x^2-1)}}\cdot[\cancel{x\,(x^2-1)})] &= \frac{A}{\cancel{x}}\cdot[\cancel{x}(x^2-1)] + \frac{B}{\cancel{x-1}}\cdot[x\cancel{(x-1)}\,(x+1)] + \frac{C}{\cancel{x+1}}\cdot[x(x-1)\cancel{(x+1)}] \\
&= A(x^2-1) + B\,x\,(x+1) + C\,x\,(x-1) \\
-6x^2+3x+5 &= Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx
\end{aligned}$$

- From the coefficients of x^2 we get $-6 = A + B + C$,
- From the coefficients of x^1 we get $3 = B - C$.
- From the coefficients of x^0 we get $5 = -A$,

  Solving the system of equations gives us $A = -5$, $B = 1$ and $C = -2$.

And the integral becomes

$$\begin{aligned}
\int \frac{-6x^2+3x+5}{x^3-x}\,dx &= \int\left(-\frac{5}{x} + \frac{1}{x-1} - \frac{2}{x+1}\right)dx \\
&= -5\int\frac{1}{x}\,dx + \int\frac{1}{x-1}\,dx - 2\int\frac{1}{x+1}\,dx \\
&= -5\ln|x| + \ln|x-1| - 2\ln|x+1| + C
\end{aligned}$$

7. $\displaystyle\int \frac{15x^2 - 11x - 5}{x\,(x+1)(2x-5)}\,dx$

$$\frac{15x^2 - 11x - 5}{x\,(x+1)(2x-5)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{2x-5}$$

Multiply both sides by the denominator of the right hand side:

$$\frac{15x^2 - 11x - 5}{x\,(x+1)(2x-5)} \cdot [x\,(x+1)(2x-5)] = \frac{A}{x} \cdot [x\,(x+1)(2x-5)] + \frac{B}{x+1} \cdot [x\,(x+1)(2x-5)] + \frac{C}{2x-5} \cdot [x\,(x+1)(2x-5)]$$

Canceling terms

$$\frac{15x^2 - 11x - 5}{\cancel{x\,(x+1)(2x-5)}} \cdot [\cancel{x\,(x+1)(2x-5)}] = \frac{A}{\cancel{x}} \cdot [\cancel{x}(x+1)(2x-5)] + \frac{B}{\cancel{x+1}} \cdot [x\cancel{(x+1)}(2x-5)] + \frac{C}{\cancel{2x-5}} \cdot [x\,(x+1)\cancel{(2x-5)}]$$

$$\begin{aligned} 15x^2 - 11x - 5 &= A(x+1)(2x-5) + B\,x\,(2x-5) + C\,x\,(x+1) \\ &= A\left(2x^2 - 3x - 5\right) + 2Bx^2 - 5Bx + Cx^2 + Cx \\ &= (2A + 2B + C)x^2 + (-3A - 5B + C)x - 5A \end{aligned}$$

- From the coefficients of x^0 we get $5 = -5A$, which gives us $A = 1$.
- From the coefficients of x^1 we get $-11 = -3A - 5B + C = -3 - 5B + C$, which gives $C = -8 + 5B$.
- From the coefficients of x^2 we get $15 = 2A + 2B + C = 2 + 2B - 8 + 5B$, which gives us $B = 3$ and $C = -8 + 5B = 7$.

And the integral becomes

$$\begin{aligned} \int \frac{15x^2 - 11x - 5}{x\,(x+1)(2x-5)}\,dx &= \int \frac{1}{x}\,dx + 3\int \frac{1}{x+1}\,dx + 7\int \frac{1}{2x-5}\,dx \\ &= \boxed{\ln|x| + 3\ln|x+1| + \frac{7}{2}\ln|2x-5| + C.} \end{aligned}$$

8. $\displaystyle\int \frac{x+26}{x^2+3x-10}\,dx$

Factor the denominator as $x^2 + 3x - 10 = (x+5)(x-2)$, so that

$$\frac{x+26}{x^2+3x-10} = \frac{x+26}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$$

Multiply both sides by the denominator of the right hand side

$$\frac{x+26}{(x+5)(x-2)} \cdot [(x+5)(x-2)] = \frac{A}{x+5} \cdot [(x+5)(x-2)] + \frac{B}{x-2} \cdot [(x+5)(x-2)]$$

Canceling terms gives

$$\begin{aligned} \frac{x+26}{\cancel{(x+5)(x-2)}} \cdot [\cancel{(x+5)(x-2)}] &= \frac{A}{\cancel{x+5}} \cdot [\cancel{(x+5)}(x-2)] + \frac{B}{\cancel{x-2}} \cdot [(x+5)\cancel{(x-2)}] \\ x + 26 &= Ax - 2A + Bx + 5B \end{aligned}$$

- From the coefficients of x^1 we get $1 = A + B$, which gives us $B = 1 - A$.
- From the coefficients of x^0 we get $26 = -2A + 5B = -2A + 5 - 5A$, which gives us $A = -3$ and $B = 4$.

And the integral becomes

$$\int \frac{x+26}{x^2+3x-10}\,dx = -3\int \frac{1}{x+5}\,dx + 4\int \frac{1}{x-2}\,dx = \boxed{-3\ln|x+5| + 4\ln|x-2| + C.}$$

9. $\displaystyle\int \frac{2x^2-9x-10}{x^2-5x}\,dx$

Since this is an improper fraction, we first need to perform long division

$$\begin{array}{r|l} & \quad 2 \\ \hline x^2-5x) & 2x^2 - 9x - 10 \\ & -2x^2 + 10x \\ \hline & \quad x - 10 \end{array} \qquad \frac{2x^2-9x-10}{x^2-5x} = 2 + \frac{x-10}{x^2-5x}$$

Factor the denominator as $x^2-5x = x(x-5)$, so that $\displaystyle\frac{x-10}{x(x-5)} = \frac{A}{x} + \frac{B}{x-5}$

Multiply both sides by the denominator of the right hand side:

$$\frac{x-10}{x(x-5)} \cdot [x(x-5)] = \frac{A}{x} \cdot [x(x-5)] + \frac{B}{x-5} \cdot [x(x-5)]$$

Canceling terms

$$\begin{aligned} \frac{x-10}{\cancel{x(x-5)}} \cdot [\cancel{x(x-5)}] &= \frac{A}{\cancel{x}} \cdot [\cancel{x}(x-5)] + \frac{B}{\cancel{x-5}} \cdot [x\cancel{(x-5)}] \\ x-10 &= Ax - 5A + Bx \end{aligned}$$

- From the coefficients of x^0 we get $-10 = -5A$, which gives us $A = 2$.
- From the coefficients of x^1 we get $1 = A + B = 2 + B$, which gives $B = -1$.

And the integral becomes

$$\int \frac{2x^2-9x-10}{x^2-5x}\,dx = \int \left(2 + \frac{2}{x} - \frac{1}{x-5}\right) dx = \boxed{2x + 2\ln|x| - \ln|x-5| + C.}$$

10. $\displaystyle\int \frac{3x+10}{x^2+9x+20}\,dx$

Factor the denominator as $x^2+9x+20 = (x+5)(x+4)$, and

$$\frac{3x+10}{x^2+9x+20} = \frac{3x+10}{(x+5)(x+4)} = \frac{A}{x+5} + \frac{B}{x+4}$$

Multiply both sides by the denominator of the right hand side:

$$\frac{3x+10}{(x+5)(x+4)} \cdot [(x+5)(x+4)] = \frac{A}{x+5} \cdot [(x+5)(x+4)] + \frac{B}{x+4} \cdot [(x+5)(x+4)]$$

Canceling terms

$$\begin{aligned} \frac{3x+10}{\cancel{(x+5)(x+4)}} \cdot [\cancel{(x+5)(x+4)}] &= \frac{A}{\cancel{x+5}} \cdot [\cancel{(x+5)}(x+4)] + \frac{B}{\cancel{x+4}} \cdot [(x+5)\cancel{(x+4)}] \\ 3x+10 &= Ax + 4A + Bx + 5B \end{aligned}$$

- From the coefficients of x^1 we get $3 = A + B$, which gives us $B = 3 - A$.
- From the coefficients of x^0 we get $10 = 4A + 5B = 4A + 15 - 5A$, which gives us $A = 5$ and $B = -2$.

And the integral becomes

$$\begin{aligned} \int \frac{3x+10}{x^2+9x+20}\,dx &= 5\int \frac{1}{x+5}\,dx - 2\int \frac{1}{x+4}\,dx \\ &= \boxed{5\ln|x+5| - 2\ln|x+4| + C.} \end{aligned}$$

11. $\displaystyle\int \frac{-4x^4-2x^3-26x^2-8x-44}{(x+1)(x^2+3)^2}\,dx$

$$\frac{-4x^4-2x^3-26x^2-8x-44}{(x+1)(x^2+3)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}$$

Multiply both sides by the denominator of the right hand side:

$$\frac{-4x^4-2x^3-26x^2-8x-44}{(x+1)(x^2+3)^2}\cdot\left[(x+1)(x^2+3)^2\right] = \frac{A}{x+1}\cdot\left[(x+1)(x^2+3)^2\right] + \frac{Bx+C}{x^2+3}\cdot\left[(x+1)(x^2+3)^2\right] + \frac{Dx+E}{(x^2+3)^2}\cdot\left[(x+1)(x^2+3)^2\right]$$

Canceling terms

$$\frac{-4x^4-2x^3-26x^2-8x-44}{\cancel{(x+1)(x^2+3)^2}}\cdot\left[\cancel{(x+1)(x^2+3)^2}\right] = \frac{A}{\cancel{x+1}}\cdot\left[\cancel{(x+1)}(x^2+3)^2\right] + \frac{Bx+C}{\cancel{x^2+3}}\cdot\left[(x+1)\cancel{(x^2+3)}(x^2+3)\right] + \frac{Dx+E}{\cancel{(x^2+3)^2}}\cdot\left[(x+1)\cancel{(x^2+3)^2}\right]$$

$$\begin{aligned}-4x^4-2x^3-26x^2-8x-44 &= A(x^4+6x^2+9)+(Bx+C)(x^3+x^2+3x+3)+(Dx+E)(x+1)\\ &= (A+B)x^4+(B+C)x^3+(6A+3B+C+D)x^2\\ &\quad +(3B+3C+D+E)x+(9A+3C+E)\end{aligned}$$

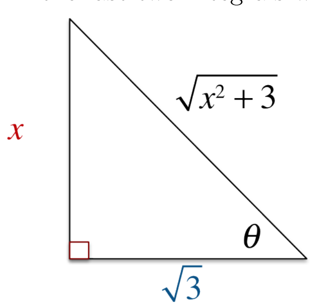
- From the coefficients of x^4 we get $-4 = A+B$, which gives us $B = -4-A$.
- From the coefficients of x^3 we get $-2 = B+C$, which gives us $C = 2+A$.
- From the coefficients of x^2 we get $-26 = 6A+3B+C+D$, which gives us $D = -16-4A$.
- From the coefficients of x^1 we get $-8 = 3B+3C+D+E$, which gives us $E = 14+4A$.
- Substituting all this on the equation for the coefficients of x^0 gives $-44 = 9A+3C+E$.

Then the solutions are $A=-4$, $B=0$, $C=-2$, $D=0$ and $E=-2$.

And the integral becomes

$$\begin{aligned}\int \frac{-4x^4-2x^3-26x^2-8x-44}{(x+1)(x^2+3)^2}\,dx &= \int\left(\frac{-4}{x+1}+\frac{-2}{x^2+3}+\frac{-2}{(x^2+3)^2}\right)dx\\ &= \int\frac{-4}{x+1}\,dx - 2\int\frac{1}{x^2+3}\,dx - 2\int\frac{1}{(x^2+3)^2}\,dx\\ &= \boxed{-4\ln|x+1| - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{1}{3}\frac{x}{x^2+3} - \frac{1}{3\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+C.}\end{aligned}$$

In the last two integrals we used the substitution $x = \sqrt{3}\tan\theta$



12. $\displaystyle\int \frac{2x^4+3x^3-8x^2-9x-10}{x\,(x^2+1)(x^2-5)}\,dx$

$$\frac{2x^4+3x^3-8x^2-9x-10}{x\,(x^2+1)(x^2-5)} = \frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{x^2-5}$$

Multiply both sides by the denominator of the right hand side:

$$\begin{aligned}\frac{2x^4+3x^3-8x^2-9x-10}{x\,(x^2+1)(x^2-5)}\cdot\left[x\,(x^2+1)(x^2-5)\right] &= \frac{A}{x}\cdot\left[x\,(x^2+1)(x^2-5)\right]+\\ &\quad \frac{Bx+C}{x^2+1}\cdot\left[x\,(x^2+1)(x^2-5)\right]+\\ &\quad \frac{Dx+E}{x^2-5}\cdot\left[x\,(x^2+1)(x^2-5)\right]\end{aligned}$$

Canceling terms

$$\begin{aligned}\frac{2x^4+3x^3-8x^2-9x-10}{\cancel{x\,(x^2+1)(x^2-5)}}\cdot\left[\cancel{x\,(x^2+1)(x^2-5)}\right] &= \frac{A}{\cancel{x}}\cdot\left[\cancel{x}(x^2+1)(x^2-5)\right]+\\ &\quad \frac{Bx+C}{\cancel{x^2+1}}\cdot\left[x\cancel{(x^2+1)}(x^2-5)\right]+\\ &\quad \frac{Dx+E}{\cancel{x^2-5}}\cdot\left[x\,(x^2+1)\cancel{(x^2-5)}\right]\end{aligned}$$

$$\begin{aligned}2x^4+3x^3-8x^2-9x-10 &= A(x^4-4x^2-5)+(Bx+C)(x^3-5x)+(Dx+E)(x^3+x)\\ &= (A+B+D)\,x^4+(C+E)\,x^3+(-4A-5B+D)\,x^2\\ &\quad +(-5C+E)\,x-5A\end{aligned}$$

- From the coefficients of x^0 we get $-10=-5A$, which gives us $A=2$.
- From the coefficients of x^1 we get $-9=-5C+E$, which gives us $E=-9+5C$.
- From the coefficients of x^3 we get $3=C+E=C-9+5C$, which gives us $C=2$ and $E=-9+5C=-9+10=1$.
- From the coefficients of x^2 we get $-8=-4A-5B+D$, which gives us $D=-8+4A+5B=-8+4\cdot 2+5B=5B$.
- From the coefficients of x^1 we get $2=A+B+D$, which gives us $2=2+B+5B$, which gives us $B=0$ and $E=0$.

And the integral becomes

$$\begin{aligned}\int \frac{2x^4+3x^3-8x^2-9x-10}{x\,(x^2+1)(x^2-5)}\,dx &= \int\left(\frac{2}{x}+\frac{2}{x^2+1}+\frac{1}{x^2-5}\right)dx\\ &= 2\int\frac{1}{x}\,dx+2\int\frac{1}{x^2+1}\,dx+\int\frac{1}{x^2-5}\,dx\\ &= 2\int\frac{1}{x}\,dx+2\int\frac{\sec^2\theta}{\sec^2\theta}\,d\theta+\int\frac{\sqrt{5}\sec\theta\tan\theta}{5\tan^2\theta}\,d\theta\\ &= \boxed{2\ln|x|+2\tan^{-1}x+\frac{\sqrt{5}}{5}\ln\left|\frac{x}{\sqrt{x^2-5}}-\frac{\sqrt{5}}{\sqrt{x^2-5}}\right|+C.}\end{aligned}$$

In the second integral we used the substitution $x=\tan\theta$ and in the third one we used the substitution $x=\sec\theta$.

13. $\displaystyle\int \frac{3x^3-7x^2+8x-1}{x\,(x-1)^3}\,dx$

$$\frac{3x^3-7x^2+8x-1}{x\,(x-1)^3} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)^3}$$

Multiply both sides by the denominator of the right hand side:

$$\frac{3x^3-7x^2+8x-1}{x\,(x-1)^3}\cdot\left[x\,(x-1)^3\right] = \frac{A}{x}\cdot\left[x\,(x-1)^3\right]+\frac{B}{x-1}\cdot\left[x\,(x-1)^3\right]+\frac{C}{(x-1)^2}\cdot\left[x\,(x-1)^3\right]+\frac{D}{(x-1)^3}\cdot\left[x\,(x-1)^3\right]$$

Canceling terms

$$\frac{3x^3-7x^2+8x-1}{\cancel{x\,(x-1)^3}}\cdot\left[\cancel{x\,(x-1)^3}\right] = \frac{A}{\cancel{x}}\cdot\left[\cancel{x}\,(x-1)^3\right]+\frac{B}{\cancel{x-1}}\cdot\left[x\,\cancel{(x-1)}(x-1)^2\right]+\frac{C}{\cancel{(x-1)^2}}\cdot\left[x\,(x-1)\cancel{(x-1)^2}\right]+\frac{D}{\cancel{(x-1)^3}}\cdot\left[x\,\cancel{(x-1)^3}\right]$$

$$\begin{aligned}3x^3-7x^2+8x-1 &= A(x^3-3x^2+3x-1)+B(x^3-2x^2+x)+C(x^2-x)+Dx\\ &= (A+B)\,x^3+(-3A+C-2B)\,x^2+(3A-C+B+D)\,x-A.\end{aligned}$$

- From the coefficients of x^0 we get $-1=-A$, which gives us $A=1$.
- From the coefficients of x^3 we get $3=A+B=1+B$, which gives us $B=2$.
- From the coefficients of x^2 we get $-7=-3A+C-2B=-3+C-4$, which gives us $C=0$.
- From the coefficients of x^1 we get $8=3A-C+B+D=3-0+2+D$, which gives us $D=3$.

And the integral becomes

$$\begin{aligned}\int \frac{3x^3-7x^2+8x-1}{x\,(x-1)^3}\,dx &= \int\left(\frac{1}{x}+\frac{2}{x-1}+\frac{3}{(x-1)^3}\right)dx\\ &= \int\frac{1}{x}\,dx+2\int\frac{1}{x-1}+3\int\frac{1}{(x-1)^3}\,dx\\ &= \boxed{\ln|x|+2\,\ln|x-1|-\frac{3}{2}\,\frac{1}{(x-1)^2}+C.}\end{aligned}$$

14. $\displaystyle\int \frac{-3x-23}{x^2-x-12}\,dx$

Factor the denominator as $x^2-x-12=(x-4)(x+3)$, so that

$$\frac{-3x-23}{x^2-x-12}=\frac{-3x-23}{(x-4)(x+3)}=\frac{A}{x-4}+\frac{B}{x+3}$$

Multiply both sides by the denominator of the right hand side and Canceling terms

$$\begin{aligned}\frac{-3x-23}{(x-4)(x+3)}\cdot\left[\cancel{(x-4)(x+3)}\right] &= \frac{A}{\cancel{x-4}}\cdot\left[\cancel{(x-4)}(x+3)\right]+\frac{B}{\cancel{x+3}}\cdot\left[(x-4)\cancel{(x+3)}\right]\\ -3x-23 &= Ax+3A+Bx-4B\end{aligned}$$

- From the coefficients of x^1 we get $-3=A+B$, which gives us $B=-3-A$.
- From the coefficients of x^0 we get $-23=3A-4B=3A+12+4A$, which gives us $A=-5$ and $B=2$.

And the integral becomes

$$\int\frac{-3x-23}{x^2-x-12}\,dx=-5\int\frac{1}{x-4}\,dx+2\int\frac{1}{x+3}\,dx=\boxed{-5\ln|x-4|+2\,\ln|x+3|+C.}$$