Integration by parts formula

$$\int u\,dv \quad = \quad u\,v - \int v\,du$$

Find the following integrals

1. $\displaystyle\int x\,e^x\,dx$

Use $u = x \;\Rightarrow\; du = dx \qquad dv = e^x\,dx \;\Rightarrow\; v = e^x$
to obtain

$$\begin{aligned}\int x\,e^x\,dx &= x\,e^x - \int e^x\,dx \\ &= x\,e^x - e^x + C.\end{aligned}$$

2. $\displaystyle\int \sqrt{x}\ln x\,dx$

Use $u = \ln x \;\Rightarrow\; du = \frac{1}{x}\,dx \qquad dv = \sqrt{x}\,dx \;\Rightarrow\; v = \frac{2}{3}x^{3/2}$

to obtain

$$\begin{aligned}\int \sqrt{x}\ln x\,dx = \int \ln x\,\sqrt{x}dx = \ln x\left(\frac{2}{3}x^{3/2}\right) - \int\left(\frac{2}{3}x^{3/2}\right)\left(\frac{1}{x}\,dx\right) &= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}\,dx \\ &= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\left(\frac{2}{3}x^{3/2}\right) + C \\ &= \frac{2}{3}x^{3/2}\left(\ln x - \frac{2}{3}\right) + C.\end{aligned}$$

3. $\displaystyle\int t\cdot 2^t\,dt$

Rewrite the integral as

$$\int t\cdot 2^t\,dt = \int t\,e^{t\ln 2}\,dt$$

Use $u = t \;\Rightarrow\; du = dt \qquad dv = e^{t\ln 2}\,dt \;\Rightarrow\; v = \frac{1}{\ln 2}\,e^{t\ln 2}$

to obtain

$$\begin{aligned}\int t\cdot 2^t\,dt = \int t\,e^{t\ln 2}\,dt = t\left(\frac{1}{\ln 2}\,e^{t\ln 2}\right) - \int\left(\frac{1}{\ln 2}\,e^{t\ln 2}\right)\,dt &= \frac{1}{\ln 2}t\cdot 2^t - \frac{1}{\ln 2}\int e^{t\ln 2}\,dt \\ &= \frac{1}{\ln 2}t\cdot 2^t - \frac{1}{\ln 2}\left(\frac{1}{\ln 2}e^{t\ln 2}\right) + C \\ &= \frac{1}{\ln 2}t\cdot 2^t - \frac{1}{(\ln 2)^2}2^t + C.\end{aligned}$$

4. $\displaystyle\int x\,\cos x\,dx$

Use $u = x \;\Rightarrow\; du = dx \qquad dv = \cos x\,dx \;\Rightarrow\; v = \sin x$
to obtain

$$\begin{aligned}\int x\cos x\,dx &= x\sin x - \int \sin x\,dx\\ &= x\,\sin x + \cos x + C.\end{aligned}$$

5. $\displaystyle\int t^2\,\cos(3t)\,dt$

Use $u = t^2 \;\Rightarrow\; du = 2t\,dt \qquad dv = \cos(3t)\,dt \;\Rightarrow\; v = \frac{1}{3}\sin(3t)$

to obtain

$$\int t^2\cos(3t)\,dt = \frac{1}{3}t^2\sin(3t) - \int \frac{2}{3}t\,\sin(3t)\,dt$$

To solve the last integral, use $u = t \;\Rightarrow\; du = dt \qquad dv = \sin(3t)\,dt \;\Rightarrow\; v = -\frac{1}{3}\cos(3t)$

to obtain

$$\begin{aligned}\frac{1}{3}t^2\,\sin(3t) - \int \frac{2}{3}t\,\sin(3t)\,dt &= \frac{1}{3}t^2\,\sin(3t) - \frac{2}{3}\left[-\frac{1}{3}t\,\cos(3t) + \frac{1}{3}\int \cos(3t)\,dt\right]\\ &= \frac{1}{3}t^2\,\sin(3t) + \frac{2}{9}t\,\cos(3t) - \frac{2}{9}\left[\frac{1}{3}\sin(3t)\right] + C\\ &= \frac{1}{3}\,t^2\,\sin(3t) + \frac{2}{9}\,t\,\cos(3t) - \frac{2}{27}\sin(3t) + C.\end{aligned}$$

6. $\displaystyle\int x\,e^{-x}\,dx$

Use $u = x \;\Rightarrow\; du = dx \qquad dv = e^{-x}\,dx \;\Rightarrow\; v = -e^{-x}$
to obtain

$$\begin{aligned}\int x\,e^{-x}\,dx = -x\,e^{-x} - \left[\int -e^{-x}\,dx\right] &= -x\,e^{-x} + \int e^{-x}\,dx\\ &= -x\,e^{-x} - e^{-x} + C.\end{aligned}$$

7. $\displaystyle\int x^3\,\ln x\,dx$

Use $u = \ln x \;\Rightarrow\; du = \frac{1}{x}\,dx \qquad dv = x^3\,dx \;\Rightarrow\; v = \frac{1}{4}x^4$

to obtain

$$\begin{aligned}\int x^3\ln x\,dx = \ln x\cdot\left(\frac{1}{4}x^4\right) - \int\left(\frac{1}{4}x^4\right)\cdot\left(\frac{1}{x}\,dx\right) &= \frac{1}{4}x^4\,\ln x - \frac{1}{4}\int x^3\,dx\\ &= \frac{1}{4}x^4\,\ln x - \frac{1}{4}\left(\frac{1}{4}x^4\right) + C = \frac{1}{4}x^4\,\ln x - \frac{1}{16}x^4 + C.\end{aligned}$$

8. $\displaystyle\int x^2\, e^{6x}\, dx$

Use $u = x^2$ $\quad dv = e^{6x}\, dx$
and tabulate

Differentiate u		Integrate v
x^2		e^{6x}
$2x$	+	$\frac{1}{6}e^{6x}$
2	-	$\frac{1}{36}e^{6x}$
0	+	$\frac{1}{216}e^{6x}$

to obtain

$$\begin{aligned}\int x^2\, e^{6x}\, dx &= (x^2)\cdot\left(\frac{1}{6}e^{6x}\right) - (2x)\cdot\left(\frac{1}{36}e^{6x}\right) + (2)\cdot\left(\frac{1}{216}e^{6x}\right) + C \\ &= \frac{1}{6}x^2\, e^{6x} - \frac{1}{18}x\, e^{6x} + \frac{1}{108}e^{6x} + C\end{aligned}$$

9. $\displaystyle\int \frac{x^2}{e^{3x}}\, dx$

Use $u = x^2$ $\quad dv = e^{-3x}\, dx$
and tabulate

Differentiate u		Integrate v
x^2		e^{-3x}
$2x$	+	$-\frac{1}{3}e^{-3x}$
2	-	$\frac{1}{9}e^{-3x}$
0	+	$-\frac{1}{27}e^{-3x}$

to obtain

$$\begin{aligned}\int x^2\, e^{6x}\, dx &= (x^2)\cdot\left(-\frac{1}{3}e^{-3x}\right) - (2x)\cdot\left(\frac{1}{9}e^{-3x}\right) + (2)\cdot\left(-\frac{1}{27}e^{-3x}\right) + C \\ &= -\frac{1}{3}x^2\, e^{-3x} - \frac{2}{9}x\, e^{-3x} - \frac{2}{27}e^{-3x} + C\end{aligned}$$

10. $\displaystyle\int_0^{\pi/4} \sin x \,\ln|\cos x|\,dx$

First we do a substitution with $z = \cos x$ and $dz = -\sin x\,dx$, to obtain

$$\int_0^{\pi/4} \sin x\,\ln|\cos x|\,dx = -\int_1^{\sqrt{2}/2} \ln|z|\,dz = \int_{\sqrt{2}/2}^{1} \ln z\,dz$$

Next we integrate by parts by using

$$u = \ln z \;\Rightarrow\; du = \frac{1}{z}\,dz \qquad dv = dz \;\Rightarrow\; v = z$$

to obtain

$$\begin{aligned}
\int_{\sqrt{2}/2}^{1} \ln z\,dz = (\ln z)\cdot(z)\Big|_{\sqrt{2}/2}^{1} - \int_{\sqrt{2}/2}^{1} (z)\cdot\left(\frac{1}{z}\,dz\right) &= z\,\ln z\Big|_{\sqrt{2}/2}^{1} - \int_{\sqrt{2}/2}^{1} dz \\
&= \left[(1)\ln(1) - \left(\frac{\sqrt{2}}{2}\right)\ln\left(\frac{\sqrt{2}}{2}\right)\right] - z\,\Big|_{\sqrt{2}/2}^{1} \\
&= \left[0 - \left(\frac{\sqrt{2}}{2}\right)\ln\left(\frac{\sqrt{2}}{2}\right)\right] - \left[1 - \frac{\sqrt{2}}{2}\right] \\
&= -\frac{\sqrt{2}}{2}\ln\left(\frac{\sqrt{2}}{2}\right) - 1 + \frac{\sqrt{2}}{2}.
\end{aligned}$$

Equivalently, we can just do integration by parts (without the initial substitution) by using

$$u = \ln|\cos x| \;\Rightarrow\; du = -\frac{1}{\cos x}\,\sin x\,dx\,dx \qquad dv = \sin x\,dx \;\Rightarrow\; v = -\cos x$$

$$\begin{aligned}
\int_0^{\pi/4} \sin x \ln|\cos x|\,dx &= (\ln|\cos x|)\,(-\cos x)\Big|_0^{\pi/4} - \int_0^{\pi/4} (-\cos x)\left(-\frac{\sin x}{\cos x}\,dx\right) \\
&= -\cos x \ln|\cos x|\Big|_0^{\pi/4} - \int_0^{\pi/4} \sin x\,dx = -\cos x \ln|\cos x|\Big|_0^{\pi/4} + \cos x\Big|_0^{\pi/4} \\
&= -\left[\cos\left(\frac{\pi}{4}\right)\ln\left|\cos\left(\frac{\pi}{4}\right)\right| - \cos(0)\ln|\cos(0)|\right] + \left[\cos\left(\frac{\pi}{4}\right) - \cos(0)\right] \\
&= -\frac{\sqrt{2}}{2}\ln\left(\frac{\sqrt{2}}{2}\right) + 0 + \frac{\sqrt{2}}{2} - 1.
\end{aligned}$$

11. $\displaystyle\int x\,\ln x\,dx$

Use

$$u = \ln x \;\Rightarrow\; du = \frac{1}{x}\,dx \qquad dv = x\,dx \;\Rightarrow\; v = \frac{1}{2}x^2$$

to obtain

$$\begin{aligned}
\int x \ln x\,dx = (\ln x)\left(\frac{1}{2}x^2\right) - \int\left(\frac{1}{2}x^2\right)\left(\frac{1}{x}\,dx\right) &= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx \\
&= \frac{1}{2}x^2\ln x - \frac{1}{2}\left(\frac{1}{2}x^2\right) + C = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C.
\end{aligned}$$

12. $\int x^3\sqrt{4-x^2}\,dx$

We recognize that can integrate $x\sqrt{4-x^2}$, as opposed to $\sqrt{4-x^2}$, then our integration by parts should be

Use $u = x^2 \;\Rightarrow\; du = 2\,x\,dx$

$dv = x\sqrt{4-x^2}\,dx \;\Rightarrow\; v = \int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int(-2x)(4-x^2)^{1/2}\,dx = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)(4-x^2)^{3/2} = -\frac{1}{3}(4-x^2)^{3/2}$

to obtain

$$\begin{aligned}\int x^2 x\,\sqrt{4-x^2}\,dx &= (x^2)\left(-\frac{1}{3}(4-x^2)^{3/2}\right) - \int\left(-\frac{1}{3}(4-x^2)^{3/2}\right)(2\,x\,dx)\\ &= -\frac{x^2}{3}(4-x^2)^{3/2} + \frac{2}{3}\int x\,(4-x^2)^{3/2}\,dx\end{aligned}$$

We use substitution on the last integral with $u = 4-x^2$ and $du = -2x\,dx$, to obtain

$$\begin{aligned}\int x^3\sqrt{4-x^2}\,dx &= -\frac{x^2}{3}(4-x^2)^{3/2} + \frac{2}{3}\int x\,(4-x^2)^{3/2}\,dx\\ &= -\frac{x^2}{3}(4-x^2)^{3/2} - \frac{1}{3}\int (u)^{3/2}\,du\\ &= -\frac{x^2}{3}(4-x^2)^{3/2} - \frac{1}{3}\left(\frac{2}{5}u^{5/2}\right) + C\\ &= -\frac{x^2}{3}(4-x^2)^{3/2} - \frac{2}{15}(4-x^2)^{5/2} + C.\end{aligned}$$

13. $\int x\,\cos^2 x\,dx$

Use the trigonometric identity $\cos^2 x = \dfrac{1+\cos(2x)}{2}$, so that

$$\int x\,\cos^2 x\,dx = \frac{1}{2}\int x\,dx + \frac{1}{2}\int x\,\cos(2x)\,dx$$

To solve the last integral use $u = x \;\Rightarrow\; du = dx \qquad dv = \cos(2x)\,dx \;\Rightarrow\; v = \frac{1}{2}\sin(2x)$

to obtain

$$\begin{aligned}\frac{1}{2}\int x\,dx + \frac{1}{2}\int x\cos(2x)\,dx &= \frac{1}{4}x^2 + \frac{1}{2}\left[x\left(\frac{1}{2}\sin(2x)\right) - \int\left(\frac{1}{2}\sin(2x)\right)dx\right]\\ &= \frac{1}{4}x^2 + \frac{1}{4}x\,\sin(2x) - \frac{1}{4}\int \sin(2x)\,dx\\ &= \frac{1}{4}x^2 + \frac{1}{4}x\,\sin(2x) - \frac{1}{4}\left(-\frac{1}{2}\cos(2x)\right) + C\\ &= \frac{1}{4}x^2 + \frac{1}{4}x\,\sin(2x) + \frac{1}{8}\cos(2x) + C.\end{aligned}$$

14. $\int e^x \sin x\, dx$

Use $u = \sin x \Rightarrow du = \cos x\, dx \qquad dv = e^x\, dx \Rightarrow v = e^x$
to obtain

$$\int e^x \sin x\, dx = (\sin x)(e^x) - \int (e^x)(\cos x\, dx) = e^x \sin x - \int e^x \cos x\, dx$$

We do integration by parts in the last integral with

$$u = \cos x \Rightarrow du = -\sin x\, dx \qquad dv = e^x\, dx \Rightarrow v = e^x$$

$$\begin{aligned} e^x \sin x - \int e^x \cos x\, dx &= e^x \sin x - \left[(\cos x)(e^x) - \int (e^x)(-\sin x\, dx)\right] \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x\, dx \end{aligned}$$

We add the last integral on both sides

$$\begin{aligned} \int e^x \sin x &= e^x \sin x - e^x \cos x - \int e^x \sin x\, dx \\ \int e^x \sin x\, dx + \int e^x \sin x\, dx &= e^x \sin x - e^x \cos x - \int e^x \sin x\, dx + \int e^x \sin x\, dx \\ 2\int e^x \sin x\, dx &= e^x \sin x - e^x \cos x \\ \int e^x \sin x\, dx &= \frac{1}{2}\left[e^x \sin x - e^x \cos x\right] + C. \end{aligned}$$

15. $\int 2x \tan^{-1} x\, dx$

Use $u = \tan^{-1} x \Rightarrow du = \frac{1}{1+x^2}\, dx \qquad dv = 2\,x\, dx \Rightarrow v = x^2$

to obtain

$$\int 2x\tan^{-1} x\, dx = (\tan^{-1} x)(x^2) - \int (x^2)\left(\frac{1}{x^2+1}\, dx\right) = x^2 \tan^{-1} x - \int \frac{x^2}{1+x^2}\, dx$$

We perform long division to simply the fraction in the last integral

$$\begin{array}{r|l} & \quad 1 \\ x^2+1 & \quad x^2 \\ & -x^2-1 \\ \hline & \quad -1 \end{array}$$

Which gives $\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$,

and

$$\begin{aligned} x^2 \tan^{-1} x - \int \frac{x^2}{1+x^2}\, dx &= x^2 \tan^{-1} x - \int \left(1 - \frac{1}{1+x^2}\right) dx \\ &= x^2 \tan^{-1} x - x + \tan^{-1} x + C. \end{aligned}$$

16. $\displaystyle\int \ln x\, dx$

Use

$$u = \ln x \;\Rightarrow\; du = \frac{1}{x}\, dx \qquad dv = dx \;\Rightarrow\; v = x$$

to obtain

$$\int \ln x\, dx = (\ln x)\,(x) - \int (x)\left(\frac{1}{x}\, dx\right) = x \ln x - \int dx = x \ln x - x + C.$$

17. $\displaystyle\int (2x+1)e^x\, dx$

Use

$$u = 2x+1 \;\Rightarrow\; du = 2\, dx \qquad dv = e^x\, dx \;\Rightarrow\; v = e^x$$

to obtain

$$\begin{aligned}\int (2x+1)e^x\, dx = (2x+1)\,(e^x) - \int (e^x)\,(2\, dx) &= (2x+1)e^x - 2\int e^x\, dx \\ &= (2x+1)e^x - 2e^x + C.\end{aligned}$$

18. $\displaystyle\int x^2\sqrt{x-1}\, dx$

Use

$$u = \sqrt{x-1} \;\Rightarrow\; du = \frac{1}{2}\,(x-1)^{-1/2}\, dx \qquad dv = x^2\, dx \;\Rightarrow\; v = \frac{1}{3}x^3$$

with this we obtain

$$\int x^2\sqrt{x-1}\, dx = (\sqrt{x-1})\left(\frac{1}{3}x^3\right) - \int \left(\frac{1}{3}x^3\right)\left(\frac{1}{2}(x-1)^{-1/2}\, dx\right) = \frac{1}{3}x^3\sqrt{x-1} - \frac{1}{6}\int \frac{x^3}{\sqrt{x-1}}\, dx$$

which is a **more** complicated integral than the original one.

That means we should change the choice of u and dv:

$$u = x^2 \;\Rightarrow\; du = 2\,x\, dx \qquad dv = \sqrt{x-1}\, dx \;\Rightarrow\; v = \frac{2}{3}(x-1)^{3/2}$$

$$\int x^2\sqrt{x-1}\, dx = (x^2)\left(\frac{2}{3}(x-1)^{3/2}\right) - \int \left(\frac{2}{3}(x-1)^{3/2}\right)(2\,x\, dx) = \frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3}\int x(x-1)^{3/2}\, dx$$

We integrate by parts one more time in the last integral with

$$u = x \;\Rightarrow\; du = dx \qquad dv = (x-1)^{3/2}\, dx \;\Rightarrow\; v = \frac{2}{5}(x-1)^{5/2}$$

$$\begin{aligned}\frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3}\int x(x-1)^{3/2}\, dx &= \frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3}\left[x\left(\frac{2}{5}(x-1)^{5/2}\right) - \int \left(\frac{2}{5}(x-1)^{5/2}\right) dx\right] \\ &= \frac{2}{3}x^2(x-1)^{3/2} - \frac{8}{15}x\,(x-1)^{5/2} + \frac{8}{15}\int (x-1)^{5/2}\, dx \\ &= \frac{2}{3}x^2(x-1)^{3/2} - \frac{8}{15}x\,(x-1)^{5/2} + \frac{8}{15}\left(\frac{2}{7}(x-1)^{7/2}\right) + C \\ &= \frac{2}{3}x^2(x-1)^{3/2} - \frac{8}{15}x\,(x-1)^{5/2} + \frac{16}{105}(x-1)^{7/2} + C.\end{aligned}$$

19. $\displaystyle\int \sin^{-1} x\, dx$

Use

$$u = \sin^{-1} x \Rightarrow du = \frac{1}{\sqrt{1-x^2}}\, dx \qquad dv = dx \Rightarrow v = x$$

to obtain

$$\int \sin^{-1} x\, dx = \left(\sin^{-1} x\right)(x) - \int (x)\left(\frac{1}{\sqrt{1-x^2}}\, dx\right) = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx$$

We solve the last integral using the substitution $u = 1 - x^2$, $du = -2x\, dx$,

$$\begin{aligned} x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx = x \sin^{-1} x - \int \left(\frac{1}{\sqrt{u}}\right)\left(-\frac{1}{2}\, du\right) &= x \sin^{-1} x + \frac{1}{2}\int u^{-1/2}\, du \\ &= x \sin^{-1} x + \frac{1}{2}\left[2u^{1/2}\right] + C \\ &= x \sin^{-1} x + u^{1/2} + C \\ &= x \sin^{-1} x + \sqrt{1-x^2} + C. \end{aligned}$$

20. $\displaystyle\int (\ln x)^2\, dx$

Use

$$u = (\ln x)^2 \Rightarrow du = \frac{2 \ln x}{x}\, dx \qquad dv = dx \Rightarrow v = x$$

to obtain

$$\begin{aligned} \int (\ln x)^2\, dx = \left((\ln x)^2\right)(x) - \int (x)\left(\frac{2 \ln x}{x}\, dx\right) &= x\, (\ln x)^2 - 2\int \ln x\, dx \\ &= (\ln x)^2 - 2\left[x \ln x - x\right] + C. \end{aligned}$$

See problem 16 for the integral of $\ln x$.

21. $\displaystyle\int \tan^{-1} x\, dx$

Use

$$u = \tan^{-1} x \Rightarrow du = \frac{1}{1+x^2}\, dx \qquad dv = dx \Rightarrow v = x$$

to obtain

$$\int \tan^{-1} x\, dx = \left(\tan^{-1} x\right)(x) - \int (x)\left(\frac{1}{1+x^2}\, dx\right) = x \tan^{-1} x - \int \frac{x}{1+x^2}\, dx$$

To solve the last integral we use the substitution $u = 1 + x^2$, $du = 2\, x\, dx$

$$\begin{aligned} \int \tan^{-1} x\, dx &= x \tan^{-1} x - \int \frac{1}{1+x^2}(x\, dx) \\ &= x \tan^{-1} x - \int \frac{1}{u}\left(\frac{1}{2}\, du\right) = x \tan^{-1} x - \frac{1}{2}\int \frac{1}{u}\, du \\ &\qquad = x \tan^{-1} x - \frac{1}{2}\left[\ln |u|\right] + C = x \tan^{-1} x - \frac{1}{2}\ln\left|1+x^2\right| + C. \end{aligned}$$

22. $\displaystyle\int x^3\sqrt{9-x^2}\,dx$

Use

$$u=\sqrt{9-x^2} \;\Rightarrow\; du=-\frac{x}{\sqrt{9-x^2}}\,dx \qquad dv=x^3\,dx \;\Rightarrow\; v=\frac{1}{4}x^4$$

to obtain

$$\begin{aligned}\int x^3\sqrt{9-x^2}\,dx &= \left(\sqrt{9-x^2}\right)\left(\frac{1}{4}x^4\right)-\int\left(\frac{1}{4}x^4\right)\left(-\frac{x}{\sqrt{9-x^2}}\,dx\right)\\ &= \frac{1}{4}x^4\sqrt{9-x^2}+\frac{1}{4}\int\frac{x^5}{\sqrt{9-x^2}}\,dx,\end{aligned}$$

To solve the last integral we use the substitution $u=9-x^2$, $x^4=(9-u)^2$, $du=-2x\,dx$:

$$\begin{aligned}\int\frac{x^5}{\sqrt{9-x^2}}\,dx=\int\frac{x^4}{\sqrt{9-x^2}}\,(x\,dx) &= \int\frac{(9-u)^2}{\sqrt{u}}\left(-\frac{1}{2}\,du\right)=-\frac{1}{2}\int\frac{(9-u)^2}{u^{1/2}}\,du\\ &= -\frac{1}{2}\int u^{-1/2}\left(81-18u+u^2\right)\,du=-\frac{1}{2}\int\left(81u^{-1/2}-18u^{1/2}+u^{3/2}\right)\,du\\ &= -\frac{1}{2}\left[162u^{1/2}-12u^{3/2}+\frac{2}{5}u^{5/2}\right]\\ &= -\frac{1}{2}\left[162(9-x^2)^{1/2}-12(9-x^2)^{3/2}+\frac{2}{5}(9-x^2)^{5/2}\right]\\ &= 6\,(9-x^2)^{3/2}-81\,(9-x^2)^{1/2}-\frac{1}{5}(9-x^2)^{5/2},\end{aligned}$$

so that

$$\int x^3\sqrt{9-x^2}\,dx=\frac{1}{4}x^4\sqrt{9-x^2}+\frac{1}{4}\left[6\,(9-x^2)^{3/2}-81\,(9-x^2)^{1/2}-\frac{1}{5}(9-x^2)^{5/2}\right]+C.$$

Note that this integral is easier to solve using substitution from the beginning, with

$$u=9-x^2 \qquad x^2=9-u \qquad du=-2\,x\,dx$$

$$\begin{aligned}\int x^3\sqrt{9-x^2}\,dx=\int x^2\sqrt{9-x^2}\,(x\,dx) &= \int(9-u)\sqrt{u}\left(-\frac{1}{2}\,du\right)\\ &= -\frac{1}{2}\int\left(9\,u^{1/2}-u^{3/2}\right)\,du\\ &= -\frac{1}{2}\left[9\left(\frac{2}{3}u^{3/2}\right)-\frac{2}{5}u^{5/2}\right]+C\\ &= -3u^{3/2}+\frac{1}{5}u^{5/2}+C\\ &= -3(9-x^2)^{3/2}+\frac{1}{5}(9-x^2)^{5/2}+C.\end{aligned}$$

23. $\int e^{2x} \sin x\, dx$

Use

$$u = \sin x \;\Rightarrow\; du = \cos x\, dx \qquad dv = e^{2x}\, dx \;\Rightarrow\; v = \frac{1}{2} e^{2x}$$

to obtain

$$\int e^{2x} \sin x\, dx = (\sin x)\left(\frac{1}{2}e^{2x}\right) - \int \left(\frac{1}{2}e^{2x}\right)(\cos x\, dx) \quad = \quad \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\, dx$$

We apply integration by parts on the last part with

$$u = \cos x \;\Rightarrow\; du = -\sin x\, dx \qquad dv = e^{2x}\, dx \;\Rightarrow\; v = \frac{1}{2} e^{2x}$$

$$\begin{aligned}
\int e^{2x}\sin x\, dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\, dx &= \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left[(\cos x)\left(\frac{1}{2}e^{2x}\right) - \int\left(\frac{1}{2}e^{2x}\right)(-\sin x\, dx)\right] \\
&= \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left[\frac{1}{2}e^{2x}\cos x + \frac{1}{2}\int e^{2x}\sin x\, dx\right] \\
&= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin x\, dx,
\end{aligned}$$

adding the last integral on both sides gives

$$\begin{aligned}
\int e^{2x}\sin x\, dx + \frac{1}{4}\int e^{2x}\sin x\, dx, &= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin x\, dx + \frac{1}{4}\int e^{2x}\sin x\, dx \\
\frac{5}{4}\int e^{2x}\sin x\, dx &= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x \\
\int e^{2x}\sin x\, dx &= \frac{4}{5}\left[\frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x\right] + C.
\end{aligned}$$

24. $\int \cos^{-1} x\, dx$

Use

$$u = \cos^{-1} x \;\Rightarrow\; du = -\frac{1}{\sqrt{1-x^2}}\, dx \qquad dv = dx \;\Rightarrow\; v = x$$

to obtain

$$\begin{aligned}
\int \cos^{-1} x\, dx &= \left(\cos^{-1} x\right)(x) - \int (x)\left(-\frac{1}{\sqrt{1-x^2}}\, dx\right) \\
&= x\cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}}\, dx.
\end{aligned}$$

To solve the last integral we use the substitution $u = 1 - x^2$, $du = -2\,x\,dx$,

$$\begin{aligned}
\int \cos^{-1} x\, dx &= x\cos^{-1} x + \int \frac{1}{\sqrt{1-x^2}}(x\, dx) \\
&= x\cos^{-1} x + \int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}\, du\right) = x\cos^{-1} x - \frac{1}{2}\int u^{-1/2}\, du \\
&= x\cos^{-1} x - \frac{1}{2}\left(2u^{1/2}\right) + C \\
&= x\cos^{-1} x - (1-x^2)^{1/2} + C.
\end{aligned}$$

25. $\displaystyle\int \frac{\ln x}{x^2}\,dx$

Use $u = \ln x \;\Rightarrow\; du = \frac{1}{x}\,dx \qquad dv = x^{-2}\,dx \;\Rightarrow\; v = -x^{-1} = -\frac{1}{x}$

to obtain

$$\int \ln x\,\frac{dx}{x^2} = (\ln x)\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right)\left(\frac{1}{x}\,dx\right) = -\frac{\ln x}{x} + \int x^{-2}\,dx = -\frac{\ln x}{x} + \left(-x^{-1}\right) + C = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

26. $\displaystyle\int \sin^3 x\,dx$

Use $u = \sin^2 x \;\Rightarrow\; du = 2\sin x\cos x\,dx \qquad dv = \sin x\,dx \;\Rightarrow\; v = -\cos x$
to obtain

$$\begin{aligned}
\int \sin^2 x\sin x\,dx &= \left(\sin^2 x\right)(-\cos x) - \int(-\cos x)\left(2\sin x\cos x\,dx\right) = -\sin^2 x\,\cos x + 2\int \sin x\cos^2 x\,dx \\
&= -\sin^2 x\,\cos x + 2\int \sin x\left(1-\sin^2 x\right)\,dx \\
\int \sin^3 x\,dx &= -\sin^2 x\,\cos x + 2\int \sin x\,dx - 2\int \sin^3 x\,dx \\
\int \sin^3 x\,dx + 2\int \sin^3 x\,dx &= -\sin^2 x\,\cos x + 2\int \sin x\,dx - 2\int \sin^3 x\,dx + 2\int \sin^3 x\,dx \\
3\int \sin^3 x\,dx &= -\sin^2 x\,\cos x + 2\int \sin x\,dx = -\sin^2 x\cos x - 2\cos x + C \\
\int \sin^3 x\,dx &= \frac{1}{3}\left[-\sin^2 x\cos x - 2\cos x\right] + C.
\end{aligned}$$

27. $\displaystyle\int x^2\ln x\,dx$

Use $u = \ln x \;\Rightarrow\; du = \frac{1}{x}\,dx \qquad dv = x^2\,dx \;\Rightarrow\; v = \frac{1}{3}x^3$

to obtain

$$\begin{aligned}
\int x^2\ln x\,dx = (\ln x)\left(\frac{1}{3}x^3\right) - \int\left(\frac{1}{3}x^3\right)\left(\frac{1}{x}\,dx\right) &= \frac{1}{3}x^3\,\ln x - \frac{1}{3}\int x^2\,dx \\
&= \frac{1}{3}x^3\,\ln x - \frac{1}{3}\left(\frac{1}{3}x^3\right) + C = \frac{1}{3}x^3\,\ln x - \frac{1}{9}x^3 + C.
\end{aligned}$$

28. $\displaystyle\int x\sin\left(\frac{x}{2}\right)\,dx$

Use $u = x \;\Rightarrow\; du = dx \qquad dv = \sin\left(\frac{x}{2}\right)\,dx \;\Rightarrow\; v = -2\cos\left(\frac{x}{2}\right)$

to obtain

$$\begin{aligned}
\int x\sin\left(\frac{x}{2}\right)\,dx = (x)\left(-2\cos\left(\frac{x}{2}\right)\right) - \int\left(-2\cos\left(\frac{x}{2}\right)\right)(dx) &= -2\,x\,\cos\left(\frac{x}{2}\right) + 2\int\cos\left(\frac{x}{2}\right)\,dx \\
&= -2\,x\,\cos\left(\frac{x}{2}\right) + 2\left[2\sin\left(\frac{x}{2}\right)\right] + C \\
&= -2\,x\,\cos\left(\frac{x}{2}\right) + 4\sin\left(\frac{x}{2}\right) + C.
\end{aligned}$$