Evaluate the following integrals

1. $\displaystyle\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{16\tan^{-1}x}{1+x^2}\,dx$

To solve the integral we use the substitution $u = \tan^{-1}x, \qquad du = \dfrac{1}{x^2+1}\,dx$

And the integral becomes

$$\int \frac{16\tan^{-1}x}{1+x^2}\,dx = \int 16u\,du = 8u^2 = 8\left(\tan^{-1}x\right)^2$$

evaluating the limits of the integral gives

$$\int_0^b \frac{16\tan^{-1}x}{1+x^2}\,dx = 8\left(\tan^{-1}x\right)^2\Big|_0^b = 8\left(\tan^{-1}(b)\right)^2 - 8\left(\tan^{-1}(0)\right)^2 = 8\left(\tan^{-1}(b)\right)^2$$

Finally, we evaluate the limit

$$\lim_{b\to\infty} 8\left(\tan^{-1}(b)\right)^2 = 8\left(\frac{\pi}{2}\right)^2 = \boxed{2\pi^2.}$$

2. $\displaystyle\int_0^2 \frac{2x}{x^2-4}\,dx$

To solve the integral we use the substitution $u = x^2-4, \qquad du = 2\,x\,dx$

And the integral becomes

$$\begin{aligned}\int_0^2 \frac{2x}{x^2-4}\,dx = \lim_{b\to 2^-}\int_0^b \frac{2x}{x^2-4}\,dx &= \lim_{b\to 2^-}\left[\ln\left|x^2-4\right|\right]_0^b \\ &= \lim_{b\to 2^-}\ln\left|b^2-4\right| - \ln(4) = \boxed{-\infty.}\end{aligned}$$

3. $\displaystyle\int_{-\infty}^\infty \frac{2x}{(x^2+1)^2}\,dx$

To solve the integral we use the substitution $u = x^2-4, \qquad du = 2\,x\,dx$

And the integral becomes

$$\int \frac{2x}{x^2-4}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} = -\frac{1}{x^2+1}.$$

And we obtain

$$\begin{aligned}\int_{-\infty}^\infty \frac{2x}{(x^2+1)^2}\,dx &= \lim_{b\to-\infty}\int_b^0 \frac{2x}{(x^2+1)^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{2x}{(x^2+1)^2}\,dx \\ &= \lim_{b\to-\infty}\left[-\frac{1}{x^2+1}\right]_b^0 + \lim_{b\to\infty}\left[-\frac{1}{x^2+1}\right]_0^b \\ &= -1 + \lim_{b\to-\infty}\left[\frac{1}{b^2+1}\right] - \lim_{b\to\infty}\left[\frac{1}{b^2+1}\right] - (-1) = \boxed{0.}\end{aligned}$$

4. $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

The discontinuity is at $x = 0$

And the integral becomes

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{b\to 0}\int_b^1 x^{-1/2}\,dx = \lim_{b\to 0}\left[2\sqrt{x}\right]_b^1 \\ &= 2 - \lim_{b\to 0} 2\sqrt{b} = \boxed{2.}\end{aligned}$$

5. $\int_0^\infty \frac{1}{x^2+1}\,dx$

To solve the integral we trigonometric substitution $x = \tan\theta, \qquad dx = \sec^2\theta\,d\theta$

And the integral becomes

$$\int \frac{1}{x^2+1}\,dx = \int \frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta = \frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \int d\theta = \theta = \tan^{-1} x$$

And we obtain

$$\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\left[\tan^{-1} x\right]_0^b \quad = \quad \lim_{b\to\infty}\tan^{-1} b - 0 = \boxed{\frac{\pi}{2}.}$$

6. $\int_0^2 \frac{h+1}{\sqrt{4-h^2}}\,dh$

- To solve the first part of the integral we use the substitution $u = 4-h^2, \qquad du = -2\,h\,dh$

$$\int \frac{h}{\sqrt{4-h^2}}\,dh = \frac{1}{2}\int u^{-1/2}\,du = -u^{1/2} = -\sqrt{4-h^2}.$$

- To solve the second part of the integral we use trigonometric substitution $h = 2\sin\theta, \qquad du = 2\cos\theta\,d\theta$

$$\int \frac{1}{\sqrt{4-h^2}}\,dh = \int \frac{2\cos\theta}{\sqrt{4-4\sin^2\theta}}\,d\theta = \int \frac{2\cos\theta}{2\cos\theta}\,d\theta = \int d\theta = \theta = \sin^{-1}\left(\frac{h}{2}\right).$$

Since the discontinuity is at $x = 2$, the problem becomes

$$\begin{aligned}\int_0^2 \frac{h+1}{\sqrt{4-h^2}}\,dh = \lim_{b\to 2^-}\int_0^b \frac{h+1}{\sqrt{4-h^2}}\,dh &= \lim_{b\to 2^-}\left[-\sqrt{4-h^2}+\sin^{-1}\left(\frac{h}{2}\right)\right]_0^b \\ &= \lim_{b\to 2^-} -\sqrt{4-b^2}+\sqrt{4-0}+\lim_{b\to 2^-}\sin^{-1}\left(\frac{b}{2}\right)-\sin^{-1}(0) \\ &= 0+2+\sin^{-1}(1)-0 = \boxed{2+\frac{\pi}{2}.}\end{aligned}$$

7. $\displaystyle\int_{-\infty}^{0} \theta e^{\theta}\, d\theta$

To solve the integral we use integration by parts $u = \theta, \quad du = d\theta, \qquad dv = e^{\theta}\, d\theta, \quad v = e^{\theta}$

$$\int \theta e^{\theta}\, d\theta = \theta \cdot e^{\theta} - \int e^{\theta}\, d\theta = \theta\, e^{\theta} - e^{\theta}$$

And the problem becomes

$$\begin{aligned}\int_{-\infty}^{0} \theta e^{\theta}\, d\theta = \lim_{b\to-\infty} \int_{b}^{0} \theta e^{\theta}\, d\theta = \lim_{b\to-\infty} \left[\theta\, e^{\theta} - e^{\theta}\right]_b^0 &= 0 - e^0 - \lim_{b\to-\infty} \left(b\, e^{b} - e^{b}\right) \\ &= -1 - (0-0) = \boxed{-1.}\end{aligned}$$

8. $\displaystyle\int_{1}^{\infty} \frac{1}{r^2}\, dr$

$$\int_{1}^{\infty} \frac{1}{r^2}\, dr = \lim_{b\to\infty} \int_{1}^{b} r^{-2}\, dr = \lim_{b\to\infty} \left[-\frac{1}{r}\right]_1^b = \left(\lim_{b\to\infty} \left[-\frac{1}{b}\right]\right) - (-1) = 0 + 1 = \boxed{1.}$$

9. $\displaystyle\int_{1}^{\infty} \frac{1}{r}\, dr$

$$\int_{1}^{\infty} \frac{1}{r}\, dr = \lim_{b\to\infty} \int_{1}^{b} \frac{1}{r}\, dr = \lim_{b\to\infty} \left[\ln|r|\right]_1^b = \left(\lim_{b\to\infty} \left[\ln|b|\right]\right) - (0) = \boxed{\infty.}$$

10. $\displaystyle\int_{1}^{\infty} \frac{1}{r^{0.5}}\, dr$

$$\int_{1}^{\infty} r^{-0.5}\, dr = \lim_{b\to\infty} \int_{1}^{b} r^{-0.5}\, dr = \lim_{b\to\infty} \left[2r^{0.5}\right]_1^b = \left(\lim_{b\to\infty} \left[2b^{0.5}\right]\right) - (2) = \boxed{\infty.}$$

11. $\displaystyle\int_{1}^{\infty} \frac{1}{r^{1.0001}}\, dr$

$$\begin{aligned}\int_{1}^{\infty} r^{-1.0001}\, dr = \lim_{b\to\infty} \int_{1}^{b} r^{-1.0001}\, dr &= \lim_{b\to\infty} \left[-\frac{1}{0.0001} r^{-0.0001}\right]_1^b = \lim_{b\to\infty} \left[-\frac{1}{0.0001} \frac{1}{r^{0.0001}}\right]_1^b \\ &= \left(\lim_{b\to\infty} \left[-\frac{1}{0.0001} \frac{1}{b^{0.0001}}\right]\right) - \left(-\frac{1}{0.0001} \cdot 1\right) = 0 + \frac{1}{0.0001} \\ &= \frac{1}{0.0001} = \boxed{1000.}\end{aligned}$$

12. $\displaystyle\int_0^\infty \frac{1}{(1+x)\sqrt{x}}\,dx$

To solve the integral we first use the substitution $u=\sqrt{x}, \quad x=u^2, \quad du=\frac{1}{2\sqrt{x}}\,dx \Rightarrow 2u\,du=dx$

$$\int \frac{1}{(1+x)\sqrt{x}}\,dx = \int \frac{1}{(1+u^2)u}\cdot(2\,u\,du) = \int \frac{2}{1+u^2}\,du$$

To solve this integral we trigonometric substitution $u=\tan\theta, \quad du=\sec^2\theta\,d\theta$

$$\begin{aligned}\int \frac{2}{1+u^2}\,du = \int \frac{2}{1+\tan^2\theta}\cdot\left(\sec^2\theta\,d\theta\right) = \int \frac{2\sec^2\theta}{\sec^2\theta}\,d\theta = 2\int d\theta &= 2\theta \\ &= 2\tan^{-1}u = 2\tan^{-1}\left(\sqrt{x}\right).\end{aligned}$$

So that

$$\begin{aligned}\int_0^\infty \frac{1}{(1+x)\sqrt{x}}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{(1+x)\sqrt{x}} = \lim_{b\to\infty}\left[2\tan^{-1}\left(\sqrt{x}\right)\right]_0^b &= \lim_{b\to\infty}\left[2\tan^{-1}\left(\sqrt{b}\right)\right]-2\tan^{-1}(0) \\ &= 2\left(\frac{\pi}{2}\right)-2(0) = \boxed{\pi.}\end{aligned}$$

13. $\displaystyle\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$

To solve the integral we use the substitution $u=4-x, \quad du=-dx$

$$\int \frac{1}{\sqrt{4-x}} = \int -\frac{1}{\sqrt{u}}\,du = -\int u^{-1/2}\,du = -2u^{1/2} = -2\sqrt{4-x}$$

Since the integrand is discontinuous at $x=4$, we have

$$\begin{aligned}\int_0^4 \frac{1}{\sqrt{4-x}}\,dx = \lim_{b\to4}\left[\int_0^b \frac{1}{\sqrt{4-x}}\,dx\right] = \lim_{b\to4}\left[-2\sqrt{4-x}\right]_0^b &= \lim_{b\to4}\left[-2\sqrt{4-b}\right]-\left(-2\sqrt{4-0}\right) \\ &= 0+2\cdot\sqrt{4} = 2\cdot2 = \boxed{4.}\end{aligned}$$

14. $\displaystyle\int_1^\infty \frac{1}{\sqrt{e^x-1}}\,dx$

To solve the integral we use the substitution $u=e^x-1, \quad du=e^x\,dx \Rightarrow \frac{1}{1+u}\,du=dx$

$$\int \frac{1}{\sqrt{e^x-1}}\,dx = \int \frac{1}{\sqrt{u}}\cdot\left(\frac{1}{1+u}\,du\right) = \int \frac{1}{(1+u)\sqrt{u}}\,du = 2\tan^{-1}(\sqrt{u}) = 2\tan^{-1}(\sqrt{e^x-1})$$

Note that the last two steps were obtained by comparing to the integral in problem **12**. Finally, we have

$$\begin{aligned}\int_1^\infty \frac{1}{\sqrt{e^x-1}}\,dx = \lim_{b\to\infty}\left[\int_1^b \frac{1}{\sqrt{e^x-1}}\,dx\right] &= \lim_{b\to\infty}\left[2\tan^{-1}\left(\sqrt{e^x-1}\right)\right]_1^b \\ &= \lim_{b\to\infty}\left[2\tan^{-1}\left(\sqrt{e^b-1}\right)\right]-\left(2\tan^{-1}\left(\sqrt{e^1-1}\right)\right) \\ &= 2\left(\frac{\pi}{2}\right)-2\tan^{-1}(\sqrt{e^1-1}) = \boxed{\pi-2\tan^{-1}\left(\sqrt{e^1-1}\right)}\end{aligned}$$

Explain why each of these integral is improper

1. $\int_0^{\infty} x^4 e^{-x^4}\, dx$

Since the integral has an infinite interval of integration, it is an improper integral of **Type I**.

2. $\int_0^{2} \frac{x}{x^2-5x+6}\, dx$

We can factor the fraction as

$$\frac{x}{x^2-5x+6} = \frac{x}{(x-2)(x+3)}$$

Since the integral has a discontinuity at $x = 2$ and $2 \in [0,2]$, it is an improper integral of **Type II**.

3. $\int_0^{\pi/2} \sec x\, dx$

Since $\sec x$ has a discontinuity at $x = \pi/2$ and this is within the interval of integration, it is an improper integral of **Type II**.

4. $\int_{-\infty}^{0} \frac{1}{x^2+5}\, dx$

Since the integral has an infinite interval of integration, it is an improper integral of **Type I**.

Test the following integrals for convergence

1. $\displaystyle\int_0^{\pi/2} \tan\theta \, d\theta$

We will show convergence by evaluating the integral

$$\int_0^{\pi/2} \tan\theta \, d\theta = \lim_{b\to\pi/2^-} \int_0^b \tan\theta \, d\theta = \lim_{b\to\pi/2^-} \Big[\ln|\sec\theta| \Big]_0^b = \lim_{b\to\pi/2^-} [\ln|\sec(b)|] - \ln|\sec(0)| = \infty - 0 = \infty.$$

The integral is divergent.

2. $\displaystyle\int_0^{\infty} \frac{x}{x^3+1} \, dx$

We will show convergence using the comparison test

Since for $x > 0$ $\qquad \dfrac{x}{x^3+1} < \dfrac{x}{x^3} = \dfrac{1}{x^2}$

and $\qquad \displaystyle\int_1^{\infty} \frac{1}{x^2} \, dx \qquad$ is convergent ($p-$integral with $p = 2 > 1$), then the integral

$$\int_1^{\infty} \frac{x}{x^3+1} \, dx$$

is convergent by the compression test.

We can separate our original integral as

$$\int_0^{\infty} \frac{x}{x^3+1} \, dx = \int_0^{1} \frac{x}{x^3+1} \, dx + \int_1^{\infty} \frac{x}{x^3+1} \, dx$$

since the first integral is proper, it is just a constant and the whole integral is convergent.

3. $\displaystyle\int_0^{1} \frac{e^{-\sqrt{x}}}{\sqrt{x}} \, dx$

We will show convergence by evaluating the integral

$$\begin{aligned} \int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}} \, dx &= \lim_{b\to 0} \left[\int_b^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}} \, dx \right] = \lim_{b\to 0} \left[\int_{\sqrt{b}}^1 \frac{e^{-u}}{u} \cdot (2u \, du) \right] \left[\begin{array}{c} u = \sqrt{x} \\ du = dx/(2\sqrt{x}) \end{array} \right] \\ &= \lim_{b\to 0} \left[\int_{\sqrt{b}}^1 2e^{-u} \, du \right] = \lim_{b\to 0} \Big[-2e^{-u} \Big]_{\sqrt{b}}^1 = \left[-2e^{-1}\right] - \lim_{b\to 0} \left[-2e^{-\sqrt{b}} \right] = 2 - 2e^{-1}. \end{aligned}$$

The integral is convergent.

4. $\displaystyle\int_1^\infty x^{-2}\, e^{-1/x}\, dx$

We will show convergence using the limit comparison test

Let $f(x) = x^{-2}\, e^{-1/x}$ and $g(x) = x^{-2}$. Since

$$\int_1^\infty \frac{1}{x^2}\, dx$$

is convergent by the $p-$test, and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{-2}\, e^{-1/x}}{x^{-2}} = \lim_{x\to\infty} e^{-1/x} = e^0 = 1$$

The integral is convergent.

5. $\displaystyle\int_\pi^\infty \frac{1+\sin x}{x^2}\, dx$

Separate the integral into two integrals and use comparison test

$$\int_\pi^\infty \frac{1+\sin x}{x^2}\, dx = \int_\pi^\infty \frac{1}{x^2}\, dx + \int_\pi^\infty \frac{\sin x}{x^2}\, dx$$

The first integral is convergent by the $p-$test. For the second integral we note that on $[\pi, \infty)$

$$\begin{aligned} -1 &\le \sin x \le 1 \\ -\frac{1}{x^2} &\le \frac{\sin x}{x^2} \le \frac{1}{x^2} \end{aligned}$$

By the comparison test, the second integral is also convergent so that the original integral is convergent.

6. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^3}\, dx$

We will show convergence by evaluating the integral

$$\begin{aligned} \int_e^\infty \frac{1}{x(\ln x)^3}\, dx = \lim_{b\to\infty}\left[\int_e^b \frac{1}{x(\ln x)^3}\, dx\right] &= \lim_{b\to\infty}\left[\int_1^{\ln b} u^{-3}\, du\right] \left[\begin{array}{c} u = \ln u \\ du = dx/x \end{array}\right] \\ &= \lim_{b\to\infty}\left[-\frac{1}{2u^2}\right]_1^{\ln b} = \lim_{b\to\infty}\left[-\frac{1}{2(\ln b)^2}\right] + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

The integral is convergent.