Find the following integrals

1. $\int (5x+4)^2\, dx$

Use the substitution to obtain

$u = 5x + 4 \qquad du = 5\, dx$

$$\int (5x+4)^2\, dx = \int u^2 \cdot \frac{1}{5}\, du = \frac{1}{5}\int u^2\, du$$
$$= \frac{1}{5}\cdot\left(\frac{1}{3}u^3 + C\right) = \frac{1}{15}u^3 + C$$
$$= \frac{1}{15}(5x+4)^3 + C.$$

2. $\int \frac{x^3+2x}{x^2+1}\, dx$

Simplify the fraction by performing long division,

$$\begin{array}{r|l} & x \\ x^2+1 & x^3+2x \\ & -x^3-x \\ & x \end{array}$$

obtaining

$$\int \frac{x^3+2x}{x^2+1}\, dx = \int \left(x + \frac{x}{x^2+1}\right) dx = \int x\, dx + \int \frac{x}{x^2+1}\, dx.$$

To solve the second integral use the substitution to obtain

$u = x^2 + 1 \qquad du = 2\,x\, dx$

$$\int x\, dx + \int \frac{x}{x^2+1}\, dx = \int x\, dx + \frac{1}{2}\int \frac{1}{u}\, du$$
$$= \frac{1}{2}x^2 + \frac{1}{2}\ln|u| + C$$
$$= \frac{1}{2}x^2 + \frac{1}{2}\ln\left|x^2+1\right| + C.$$

3. $\int 3t^2\left(t^3+4\right)^5\, dt$

Use the substitution to obtain

$u = t^3 + 4 \qquad du = 3t^2\, dt$

$$\int 3t^2\left(t^3+4\right)^5\, dt = \int \left(t^3+4\right)^5 \cdot \left(3t^2\, dt\right)$$
$$= \int u^5\, du$$
$$= \frac{1}{6}u^6 + C$$
$$= \frac{1}{6}\left(t^3+4\right)^6 + C.$$

4. $\int \sqrt{4x-5}\,dx$

Use the substitution to obtain $u = 4x-5 \qquad du = 4\,dx$

$$\int \sqrt{4x-5}\,dx = \int \sqrt{u}\cdot\left(\frac{1}{4}\,du\right) = \frac{1}{4}\int u^{1/2}\,du$$
$$= \frac{1}{4}\left(\frac{2}{3}u^{3/2}\right) + C = \frac{1}{6}u^{3/2} + C$$
$$= \frac{1}{6}(4x-5)^{3/2} + C.$$

5. $\int t^2\left(t^3+4\right)^{-1/2}\,dt$

Use the substitution to obtain $u = t^3+4 \qquad du = 3\,t^2\,dt$

$$\int t^2\left(t^3+4\right)^{-1/2}\,dt = \int \left(t^3+4\right)^{-1/2}\cdot\left(t^2\,dt\right)$$
$$= \int u^{-1/2}\cdot\left(\frac{1}{3}\,du\right) = \int \frac{1}{3}u^{-1/2}\,du$$
$$= \frac{1}{3}\cdot\left(2u^{1/2}\right) + C = \frac{2}{3}u^{1/2} + C$$
$$= \frac{2}{3}\left(t^3+4\right)^{1/2} + C.$$

6. $\int \cos(2x+1)\,dx$

Use the substitution to obtain $u = 2x+1 \qquad du = 2\,dx$

$$\int \cos(2x+1)\,dx = \cos(u)\cdot\left(\frac{1}{2}\,du\right)$$
$$= \frac{1}{2}\int \cos(u)\,du$$
$$= \frac{1}{2}\sin(u) + C$$
$$= \frac{1}{2}\sin(2x+1) + C.$$

7. $\int 4xe^{x^2+1}\,dx$

Use the substitution to obtain $u = x^2+1 \qquad du = 2\,x\,dx$

$$\int 4xe^{x^2+1}\,dx = 2\int e^{x^2+1}\cdot(2\,x\,dx)$$
$$= 2\int e^u\,du = 2e^u + C = 2e^{x^2+1} + C.$$

8. $\displaystyle\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}\,dx$

Use the substitution $u = \sqrt{x} - 1 \qquad du = \dfrac{1}{2\sqrt{x}}\,dx$

to obtain

$$\begin{aligned}\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}\,dx &= \int (\sqrt{x}-1)^2 \cdot \left(\frac{1}{\sqrt{x}}\,dx\right) \\ &= \int u^2 \cdot (2\,du) = 2\int u^2\,du \\ &= \frac{2}{3}u^3 + C \\ &= \frac{2}{3}(\sqrt{x}-1)^3 + C.\end{aligned}$$

9. $\displaystyle\int \sqrt{x^3+x^2}\,\left(3x^2+2x\right)\,dx$

Use the substitution $u = x^3 + x^2 \qquad du = \left(3x^2+2\right)\,dx$

to obtain

$$\begin{aligned}\int \sqrt{x^3+x^2}\,\left(3x^2+2x\right)\,dx = \int \sqrt{u}\,du &= \int u^{1/2}\,du \\ &= \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{3}\left(x^3+x^2\right)^{3/2} + C.\end{aligned}$$

10. $\displaystyle\int_0^1 \frac{x+1}{(x^2+2x+2)^3}\,dx$

Use the substitution $u = x^2 + 2x + 2 \qquad du = (2x+2)\,du = 2\,(x+1)\,dx$

when $x = 0$, $u = 2$ and when $x = 1$, $u = 5$, which gives us

$$\begin{aligned}\int_0^1 \frac{x+1}{(x^2+2x+2)^3}\,dx &= \int_0^1 \frac{1}{(x^2+2x+2)^3} \cdot (x+1)\,dx \\ &= \int_2^5 \frac{1}{u^3} \cdot \frac{1}{2}\,du = \frac{1}{2}\int_2^5 u^{-3}\,du \\ &= \frac{1}{2} \cdot \left(\frac{u^{-2}}{-2}\right)\Bigg|_2^5 = -\frac{1}{4}u^{-2}\Bigg|_2^5 = -\frac{1}{4u^2}\Bigg|_2^5 \\ &= \left[-\frac{1}{4(5)^2}\right] - \left[-\frac{1}{4(2)^2}\right] \\ &= -\frac{1}{100} + \frac{1}{16} \\ &= \frac{21}{400}\end{aligned}$$

11. $\int (x+1)\sin\left(x^2+2x+1\right)\,dx$

Use the substitution $u = x^2+2x+1 \qquad du = (2x+2)\,dx = 2\,(x+1)\,dx$
to obtain

$$\begin{aligned}\int (x+1)\sin\left(x^2+2x+1\right)\,dx &= \int \sin\left(x^2+2x+1\right)\cdot(x+1)\,dx \\ &= \int \sin u\cdot\frac{1}{2}du = \frac{1}{2}\int \sin u\,du \\ &= -\frac{1}{2}\cos u + C \\ &= -\frac{1}{2}\cos\left(x^2+2x+1\right)+C.\end{aligned}$$

12. $\int \left(1+\frac{1}{x}\right)^3 \frac{1}{x^2}\,dx$

Use the substitution $u = 1+\frac{1}{x} \qquad du = -\frac{1}{x^2}\,dx$

to obtain

$$\begin{aligned}\int \left(1+\frac{1}{x}\right)^3\frac{1}{x^2}\,dx = \int u^3\,(-du) &= -\int u^3\,du \\ &= -\frac{1}{4}u^4 + C \\ &= -\frac{1}{4}\left(1+\frac{1}{x}\right)^4+C.\end{aligned}$$

13. $\int x^2\sqrt{x^3+1}\,dx$

Use the substitution $u = x^3+1 \qquad du = 3\,x^2\,dx$
to obtain

$$\begin{aligned}\int x^2\sqrt{x^3+1}\,dx &= \int \sqrt{x^3+1}\cdot\left(x^2\,dx\right) \\ &= \int \sqrt{u}\cdot\left(\frac{1}{3}\,du\right) = \frac{1}{3}\int\sqrt{u}\,du = \frac{1}{3}\int u^{1/2}\,du \\ &= \frac{1}{3}\cdot\left(\frac{2}{3}u^{3/2}\right)+C = \frac{2}{9}u^{3/2}+C \\ &= \frac{2}{9}\left(x^3+1\right)^{3/2}+C.\end{aligned}$$

14. $\displaystyle\int \frac{2}{\sqrt{4x-7}}\,dx$

Use the substitution $u = 4x - 7 \qquad du = 4\,dx$
to obtain

$$\int \frac{2}{\sqrt{4x-7}}\,dx = 2\int \frac{1}{\sqrt{u}} \cdot \left(\frac{1}{4}\,du\right) = \frac{1}{2}\int u^{-1/2}\,du$$
$$= \frac{1}{2}\cdot\left(2\,u^{1/2}\right) + C = u^{1/2} + C$$
$$= \sqrt{4x-7} + C.$$

15. $\displaystyle\int \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}\,dx$

Use the substitution $u = \sqrt{x}+1 \qquad du = \dfrac{1}{2\sqrt{x}}\,dx$

to obtain

$$\int \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}\,dx = \int \frac{1}{(\sqrt{x}+1)^2}\cdot\left(\frac{1}{\sqrt{x}}\,dx\right)$$
$$= \int \left(\frac{1}{u^2}\right)\cdot(2\,du) \qquad = 2\int u^{-2}\,du$$
$$= 2\left(\frac{u^{-1}}{-1}\right) + C = -\frac{2}{u} + C$$
$$= -\frac{2}{\sqrt{x}+1} + C.$$

16. $\displaystyle\int \frac{4x^2-8x+5}{2x-3}\,dx$

Simplify the fraction by performing long division,

$$\begin{array}{r|l} & 2x - 1 \\ \hline 2x-3 & 4x^2 - 8x + 5 \\ & -4x^2 + 6x \\ \hline & -2x + 5 \\ & 2x - 3 \\ \hline & 2 \end{array}$$

obtaining

$$\int \frac{4x^2-8x+5}{2x-3}\,dx = \int \left(2x - 1 + \frac{2}{2x-3}\right)\,dx = 2\int x\,dx - \int dx + \int \frac{2}{2x-3}\,dx$$

To solve the last integral use the substitution $u = 2x - 3 \qquad du = 2\,dx$
to obtain

$$\int \left(2x - 1 + \frac{2}{2x-3}\right)\,dx = \int 2\,x\,dx - \int dx + \int \frac{2}{2x-3}\,dx = \int 2\,x\,dx - \int dx + \int \frac{1}{u}\,du$$
$$= x^2 - x + \ln|u| + C$$
$$= x^2 - x + \ln|2x-3| + C.$$

17. $\displaystyle\int_{-1}^{1} \frac{x}{\sqrt{x+1}}\,dx$

Use the substitution $u = x+1 \qquad du = dx$

when $x=-1$, $u=0$ and when $x=1$, $u=2$, which gives us

$$\begin{aligned}\int_{-1}^{1} \frac{x}{\sqrt{x+1}}\,dx = \int_0^2 \frac{u-1}{\sqrt{u}}\,du = \int_0^2 (u-1)\,u^{-1/2}\,du &= \int_0^2 u^{1/2}\,du - \int_0^2 u^{-1/2}\,du \\ &= \frac{2}{3}u^{3/2}\Big|_0^2 - 2u^{1/2}\Big|_0^2 \\ &= \frac{2}{3}\left[2^{3/2}-0\right] - 2\left[2^{1/2}-0\right] = \frac{2}{3}\left(2\sqrt{2}\right) - 2\sqrt{2} = \frac{2}{3}\sqrt{2}.\end{aligned}$$

18. $\displaystyle\int x\sqrt{3x+2}\,dx$

Use the substitution $u = 3x+2, \;\Rightarrow\; x = \dfrac{u-2}{3} \qquad du = 3\,dx$

to obtain

$$\begin{aligned}\int x\sqrt{3x+2}\,dx = \int \left(\frac{u-2}{3}\right)\cdot\left(\sqrt{u}\right)\cdot\left(\frac{1}{3}\,du\right) &= \frac{1}{9}\int \left(u^{3/2} - 2u^{1/2}\right)\,du \\ &= \frac{1}{9}\left[\frac{2}{5}u^{5/2} - 2\left(\frac{2}{3}u^{3/2}\right)\right] + C \\ &= \frac{2}{45}u^{5/2} - \frac{4}{27}u^{3/2} + C \\ &= \frac{2}{45}(3x+2)^{5/2} - \frac{4}{27}(3x+2)^{3/2} + C.\end{aligned}$$

19. $\displaystyle\int \sqrt{x}\sqrt{x\sqrt{x}+1}\,dx$

We can rewrite the integral as

$$\int x^{1/2}\sqrt{x^{3/2}+1}\,dx.$$

Use the substitution $u = x^{3/2}+1 \qquad du = \dfrac{3}{2}x^{1/2}\,dx$

to obtain

$$\begin{aligned}\int x^{1/2}\sqrt{x^{3/2}+1}\,dx = \frac{2}{3}\int \sqrt{u}\,du &= \frac{2}{3}\int u^{1/2}du \\ &= \frac{2}{3}\left(\frac{2}{3}u^{3/2}\right) + C = \frac{4}{9}u^{3/2} + C \\ &= \frac{4}{9}\left(x^{3/2}+1\right)^{3/2} + C.\end{aligned}$$

20. $\int x^3\sqrt{x^2+7}\,dx$

Use the substitution $u = x^2+7 \Rightarrow x^2 = u-7 \qquad du = 2x\,dx$

$$\int x^3\sqrt{x^2+7}\,dx = \int x^2\sqrt{x^2+7}\,(x\,dx) = \int (u-7)\sqrt{u}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int \left(u^{3/2}-7u^{1/2}\right)\,du$$
$$= \frac{1}{2}\left(\frac{2}{5}u^{5/2}-\frac{14}{3}u^{3/2}\right)+C$$
$$= \frac{1}{5}u^{5/2} - \frac{7}{3}u^{3/2} + C$$
$$= \frac{1}{5}\left(x^2+7\right)^{5/2}-\frac{7}{3}\left(x^2+7\right)^{3/2}+C.$$

21. $\int \left(x^2+1\right)\sqrt{x-2}\,dx$

Use the substitution $u = x-2 \Rightarrow x = u+2 \qquad du = dx$

$$\int \left(x^2+1\right)\sqrt{x-2}\,dx = \int \left(u^2+2u+4+1\right)\sqrt{u}\,du = \int \left(u^{5/2}+4u^{3/2}+5u^{1/2}\right)\,du$$
$$= \frac{2}{7}u^{7/2} + \frac{8}{5}u^{5/2} + \frac{10}{3}u^{3/2} + C$$
$$= \frac{2}{7}(x-2)^{7/2}+\frac{8}{5}(x-2)^{5/2}+\frac{10}{3}(x-2)^{3/2}+C.$$

22. $\int \frac{e^x}{e^x+1}\,dx$

Use the substitution $u = e^x+1 \qquad du = e^x\,dx$

$$\int \frac{1}{e^x+1}\,(e^x dx) = \int \frac{1}{u}\,du$$
$$= \ln|u| + C$$
$$= \ln|e^x+1| + C.$$

23. $\int 3x^3\left(x^2+4\right)^5\,dx$

Use the substitution $u = x^2+4 \;\Rightarrow\; x^2 = u-4 \qquad du = 2x\,dx$

$$\int 3x^3\left(x^2+4\right)^5\,dx = \int 3x^2\left(x^2+4\right)^5(x\,dx) = \int 3(u-4)u^5\left(\frac{1}{2}du\right) = \frac{3}{2}\int\left(u^6-4u^5\right)du$$
$$= \frac{3}{2}\left(\frac{1}{7}u^7-\frac{4}{6}u^6\right)+C$$
$$= \frac{3}{14}\left(x^2+4\right)^7-\left(x^2+4\right)^6+C.$$

24. $\int \frac{x^2+2x}{x^2+2x+1}\,dx$

First, simplify by using ling division

$$\begin{array}{r|l} & \quad 1 \\ x^2+2x+1 & \quad x^2+2x \\ & -x^2-2x-1 \\ \hline & \quad -1 \end{array}$$

obtaining

$$\int \frac{x^2+2x}{x^2+2x+1}\,dx = \int\left(1-\frac{1}{x^2+2x+1}\right)dx = \int\left(1-\frac{1}{(x+1)^2}\right)dx = \int dx - \int \frac{1}{(x+1)^2}dx$$

To solve the second integral use the substitution $u = x+1 \qquad du = dx$

$$\int dx - \int\frac{1}{(x+1)^2}dx = \int dx - \int u^{-2}\,du$$
$$= x - \left(-u^{-1}\right)+C = x + u^{-1}+C$$
$$= x + \frac{1}{x+1}+C.$$

25. $\int \frac{1}{x^2+8x+16}\,dx$

Rewrite $\int \frac{1}{x^2+8x+16}\,dx = \int \frac{1}{(x+4)^2}\,dx$

Use the substitution $u = x+4 \qquad du = dx$

$$\int \frac{1}{(x+4)^2}\,dx = \int u^{-2}\,du$$
$$= -u^{-1}+C = -(x+4)^{-1}+C = -\frac{1}{x+4}+C.$$

26. $\displaystyle\int \frac{3x^2+6x+2}{2x^2}\,dx$

Rewrite $\displaystyle\int \frac{3x^2+6x+2}{2x^2}\,dx = \int \left(\frac{3}{2}+3x^{-1}+x^{-2}\right)\,dx$

There is no need for a substitution

$$\int \left(\frac{3}{2}+3x^{-1}+x^{-2}\right)\,dx = \frac{3}{2}x + 3\ln|x| - \frac{1}{x} + C.$$

27. $\displaystyle\int \frac{x^2}{(x-2)^6}\,dx$

Use the substitution $u = x-2 \;\Rightarrow\; x = u+2 \qquad du = dx$

$$\begin{aligned}\int \frac{x^2}{(x-2)^6}\,dx = \int \frac{(u+2)^2}{u^6}\,du = \int \frac{u^2+4u+4}{u^6}\,du &= \int \left(u^{-4}+4u^{-5}+4u^{-6}\right)\,du \\ &= -\frac{1}{3}u^{-3} - \frac{4}{4}u^{-4} - \frac{4}{5}u^{-5} + C \\ &= -\frac{1}{3}(x-2)^{-3} - (x-2)^{-4} - \frac{4}{5}(x-2)^{-5} + C \\ &= -\frac{1}{3(x-2)^3} - \frac{1}{(x-2)^4} - \frac{4}{5(x-2)^5} + C.\end{aligned}$$

28. $\displaystyle\int \frac{(5+\ln x)^5}{x}\,dx$

Use the substitution $u = 5+\ln x \qquad du = \dfrac{1}{x}\,dx$

$$\begin{aligned}\int \frac{(5+\ln x)^5}{x}\,dx &= \int u^5\,du \\ &= \frac{1}{6}u^6 + C \\ &= \frac{1}{6}(5+\ln x)^6 + C.\end{aligned}$$