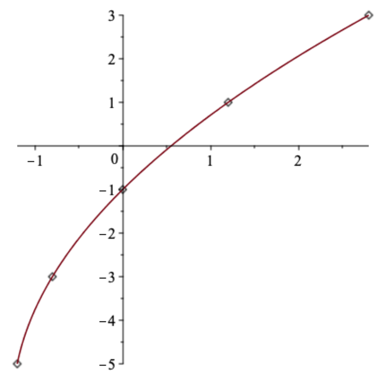
1. Consider the parametric equations

$$x = \frac{1}{5}t^2 + t, \qquad y = 2t - 1$$

(a) Sketch the parametric curve for $-2 \leq t \leq 2$

t	x	y
-2	$\frac{4}{5} - 2 = \frac{4}{5} - \frac{10}{5} = -\mathbf{\frac{6}{5}}$	$-4 - 1 = -\mathbf{5}$
-1	$\frac{1}{5} - 1 = -\mathbf{\frac{4}{5}}$	$-2 - 1 = -\mathbf{3}$
0	0	-1
1	$\frac{1}{5} + 1 = \mathbf{\frac{6}{5}}$	$2 - 1 = \mathbf{1}$
2	$\frac{4}{5} + 2 = \frac{4}{5} + \frac{10}{5} = \mathbf{\frac{14}{5}}$	$4 - 1 = \mathbf{3}$



(b) Eliminate the parameter from the set of equations.

$$\begin{aligned} t &= \frac{1}{2}(y+1) \\ &\Rightarrow \\ x &= \frac{1}{5}\left(\frac{1}{2}(y+1)\right)^2 + \frac{1}{2}(y+1) \\ &= \frac{1}{5} \times \frac{1}{4}\left(y^2 + 2y + 1\right) + \frac{1}{2}y + \frac{1}{2} \\ &= \frac{1}{20}y^2 + \frac{1}{10}y + \frac{1}{20} + \frac{1}{2}y + \frac{1}{2} \\ &= \frac{1}{20}y^2 + \frac{3}{5}y + \frac{11}{20} \end{aligned}$$

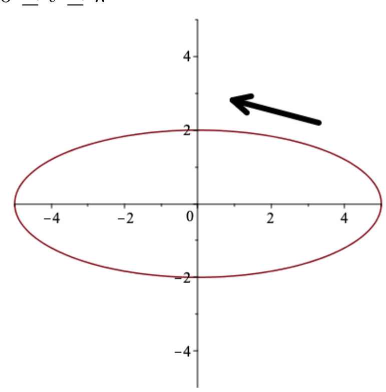
The resulting equation is

$$x = \frac{1}{20}y^2 + \frac{3}{5}y + \frac{11}{20}.$$

2. Sketch the parametric curve and clearly define the direction of motion

$$x = 5\cos(2t), \qquad y = 2\sin(2t), \qquad 0 \leq t \leq \pi$$

t	x	y
0	$5\cos(0) = \mathbf{5}$	$2\sin(0) = \mathbf{0}$
$\frac{\pi}{4}$	$5\cos\left(\frac{\pi}{2}\right) = \mathbf{0}$	$2\sin\left(\frac{\pi}{2}\right) = \mathbf{2}$
$\frac{\pi}{2}$	$5\cos(\pi) = -\mathbf{5}$	$2\sin(\pi) = \mathbf{0}$
$\frac{3\pi}{4}$	$5\cos\left(\frac{3\pi}{2}\right) = \mathbf{0}$	$2\sin\left(\frac{3\pi}{2}\right) = -\mathbf{2}$
π	$5\cos(2\pi) = \mathbf{5}$	$2\sin(2\pi) = \mathbf{0}$



3. Given the parametric curve

$$x = t^2 - 2t, \qquad y = t^3 - 2$$

(a) Find the equation of the tangent to the curve when $t = -2$.

To find the equation of the tangent line we need a point and the slope. To find the point we just need to substitute the given value of t in the equations for x and y,

$$\begin{aligned} x &= t^2 - 2t = 4 + 4 = 8 \\ y &= t^3 - 2 = -8 - 2 = -10 \end{aligned}$$

The point is $(8, -10)$.

To fin the slope we need to first find $\frac{dy}{dx}$ and then finds its value at the given point.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2}{2t-2}$$

For the given point $(t = -2)$, this becomes,

$$\frac{dy}{dx} = \frac{3t^2}{2t-2} = \frac{12}{-4-2} = \frac{12}{-6} = -2$$

The equation for the tangent line is

$$\begin{aligned} y - (-10) &= -2\,(x - 8) \\ y &= -2x + 16 - 10 = -2x + 6. \end{aligned}$$

(b) Find the point on the parametric curve where the tangent is horizontal.

At this point the slope of the tangent line is zero, so that

$$\frac{dy}{dx} = \frac{3t^2}{2t-2} = 0,$$

solving for t gives

$$\begin{aligned} \frac{3t^2}{2t-2} &= 0 \\ 3t^2 &= 0 \\ t^2 &= 0 \end{aligned}$$

This means the tangent is horizontal when $t = 0$, which gives the point $(0, -2)$.

(c) Does the parametric curve have a vertical tangent?

For a vertical tangent the slope should be infinity, so we look for values where the equation for the slope blows up, that is

$$2t - 2 = 0,$$

which gives $t = 1$ and the point $(-1, -1)$.

4. Consider the curve $\mathcal{C}$ defined by the parametric equations

$$x = t\,\cos t, \qquad y = t\,\sin t, \qquad -\pi \le t \le \pi$$

(a) Find the equation of both tangents to $\mathcal{C}$ at $\left(0, \frac{\pi}{2}\right)$.

Here we are given the point in the cartesian plane, we need to find the corresponding value of t

$$\begin{aligned} x &= t\,\cos t = 0 \\ y &= t\,\sin t = \frac{\pi}{2} \end{aligned}$$

Since we are looking for values of t that give $x = 0$ and $y = \pi/2$, we first look at the x equation. We note that $x = 0$ either if $\cos t = 0$, i.e. $t = \pm\pi/2$ or $t = 0$. When $t = 0$, $y = 0$ and that doesn't correspond to the given point. when $t = \pi/2$, $y = \pi/2$, which corresponds to the given point. Finally, when $t = -\pi/2$, $y = \pi/2$, which also corresponds to the given point. This means we need to find the slope for $t = \pi/2$ and $t = -\pi/2$.
Next we calculate the slope

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t\cos t + \sin t}{-t\sin t + \cos t}$$

substituting $t = \pi/2$ gives

$$\frac{dy}{dx} = \frac{t\cos t + \sin t}{-t\sin t + \cos t} = \frac{0+1}{-\pi/2+0} = -\frac{2}{\pi}$$

and substituting $t = -\pi/2$ gives

$$\frac{dy}{dx} = \frac{t\cos t + \sin t}{-t\sin t + \cos t} = \frac{0-1}{-\pi/2+0} = \frac{2}{\pi}$$

The two equations for the tangent lines are

$$y - \frac{\pi}{2} = -\frac{2}{\pi}x, \qquad \text{and} \qquad y - \frac{\pi}{2} = \frac{2}{\pi}x.$$

5. Find the area under the curve

$$x = 2\,\cos t, \qquad y = 3\,\sin t, \qquad 0 \le t \le \frac{\pi}{2}$$

The area under a parametric curve is given by

$$A = \left| \int_a^b g(t) f'(t) dt \right|,$$

where $x = f(t)$, $y = g(t)$ and $a \le t \le b$.
In our case

$$f(t) \;=\; 2\cos t \qquad \Rightarrow \qquad f'(t) = -2\sin t$$

and

$$\begin{aligned} A = \left| \int_0^{\pi/2} (3\sin\theta)\,(-2\sin t)\,dt \right| &= \left| -6\int_0^{\pi/2} \sin^2 t\,dt \right| = 6\left| \int_0^{\pi/2} \frac{1-\cos(2t)}{2}\,dt \right| = 3\left| \left[t - \frac{\sin(2t)}{2} \right]_0^{\pi/2} \right| \\ &= 3\left| \frac{\pi}{2} - \frac{\sin\pi}{2} - 0 + \frac{\sin 0}{2} \right| = 3\left| \frac{\pi}{2} \right| \\ &= \frac{3\pi}{2}. \end{aligned}$$

6. Find the arc length of the circle defined by

$$x = \cos 2t, \qquad y = \sin 2t, \qquad 0 \le t \le 2\pi$$

The arc length is given by

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(-2\sin(2t))^2 + (2\cos(2t))^2}\, dt = \int_0^{2\pi} \sqrt{4\sin^2(2t) + 4\cos^2(2t)}\, dt \\ &= \int_0^{2\pi} \sqrt{4\left(\sin^2(2t) + \cos^2(2t)\right)}\, dt = \int_0^{2\pi} \sqrt{4(1)}\, dt = \int_0^{2\pi} 2\, dt \\ &= 2t\big|_0^{2\pi} = 2(2\pi) - 0 = 4\pi. \end{aligned}$$

7. Find the arc length of the spiral defined by

$$x = e^t \cos t, \qquad y = e^t \sin t, \qquad 0 \le t \le 2\pi$$

The arc length is given by

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(e^t\cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2}\, dt = \int_0^{2\pi} \sqrt{e^{2t}\left[(\cos t - \sin t)^2 + (\cos t + \sin t)^2\right]}\, dt \\ &= \int_0^{2\pi} \sqrt{e^{2t}\left[\cos^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\cos t \sin t + \sin^2 t\right]}\, dt \\ &= \int_0^{2\pi} \sqrt{e^{2t}\left[\cos^2 t + \sin^2 t + \cos^2 t + \sin^2 t\right]}\, dt = \int_0^{2\pi} \sqrt{e^{2t}\,[2]}\, dt = \int_0^{2\pi} e^t\sqrt{2}\, dt \\ &= \sqrt{2}\left[e^t\right]_0^{2\pi} = \sqrt{2}\left[e^{2\pi} - e^0\right] = \sqrt{2}\left[e^{2\pi} - 1\right]. \end{aligned}$$

8. Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the given set of parametric equations

(a) $x = 7t^2 - 9t, \qquad y = t^6 + 2t^2$

$$\frac{dy}{dx} = y' = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^5 + 4t}{14t - 9}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{(14t-9)\cdot\left(30t^4+4\right) - \left(6t^5+4t\right)\cdot(14)}{(14t-9)^2}}{14t-9} = \frac{(14t-9)\cdot\left(30t^4+4\right) - 14\left(6t^5+4t\right)}{(14t-9)^3}.$$

8. Compute $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for the given set of parametric equations

(b) $x = \ln\left(3t^2\right) + 8t, \qquad y = \ln\left(t^4\right) - 6\ln\left(t^2\right)$

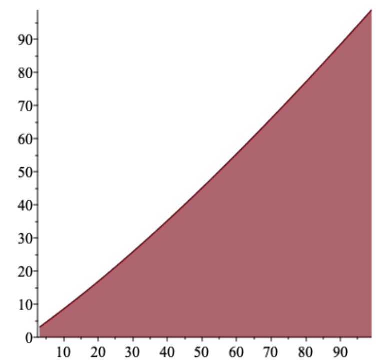
$$\frac{dy}{dx} = y' \quad = \quad \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{4t^3}{t^4} - \frac{12t}{t^2}}{\frac{6t}{3t^2} + 8} = \frac{\frac{4}{t} - \frac{12}{t}}{\frac{2}{t} + 8} = \frac{4 - 12}{2 + 8t} = -\frac{8}{2 + 8t} = -8\,(2 + 8t)^{-1}.$$

$$\frac{d^2y}{dx^2} \quad = \quad \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{8\,(2+8t)^{-2}\cdot(8)}{\frac{2}{t} + 8} = \frac{64\,t\,(2+8t)^{-2}}{2 + 8t} = \frac{64}{(2+8t)^3}.$$

9. Determine the area of the region below the parametric curve

$$x = 4t^3 - t^2, \qquad y = t^4 + 2t^2, \qquad 1 \le t \le 3$$

t	x	y
1	3	3
2	28	24
3	99	99



In this case the area is given by

$$\begin{aligned} A \quad &= \quad \int_1^3 y\,dx = \int_1^3 g(t)\cdot f'(t)\,dt = \int_1^3 \left(t^4 + 2t^2\right)\cdot\left(12t^2 - 2t\right)\,dt \\ &= \quad \int_1^3 \left(12t^6 - 2t^5 + 24t^4 - 4t^3\right)\,dt = \left[\frac{12}{7}\,t^7 - \frac{1}{3}\,t^6 + \frac{24}{5}\,t^5 - t^4\right]_1^3 \\ &= \quad \frac{12}{7}\cdot\left(3^7 - 1\right) - \frac{1}{3}\cdot\left(3^6 - 1\right) + \frac{24}{5}\cdot\left(3^5 - 1\right) - \left(3^4 - 1\right). \end{aligned}$$

10. Set up, **but do not evaluate**, an integral that gives the length of the parametric curve

(a) $x = 2 + t^2, \qquad y = e^t\,\sin(2t), \qquad 0 \le t \le 3$

$$\begin{aligned} L \quad &= \quad \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^3 \sqrt{(2t)^2 + \left(e^t\sin(2t) + 2\,e^t\,\cos(2t)\right)^2}\,dt \\ &= \quad \int_0^3 \sqrt{4t^2 + e^{2t}\sin^2(2t) + 4\,t\,e^{2t}\sin(2t)\cos(2t) + 4\,e^{2t}\cos^2(2t)}\,dt. \end{aligned}$$

10. Set up, **but do not evaluate**, an integral that gives the length of the parametric curve

(b) $x = \cos^3(2t)\,, \qquad y = \sin\left(1 - t^2\right), \qquad -\dfrac{3}{2} \leq t \leq 0$

$$\begin{aligned} L &= \int_{-3/2}^{0} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{-3/2}^{0} \sqrt{\left(-6\cos^2(2t)\sin(2t)\right)^2 + \left(-2t\cos(1-t^2)\right)^2}\, dt \\ &= \int_{-3/2}^{0} \sqrt{36\,\cos^4(2t)\,\sin^2(2t) + 4\,t^2\,\cos^2(1-t^2)}\, dt. \end{aligned}$$

11. Find the length of the parametric curve described by the parametric equations

$$x = \frac{1}{3}t^{3/2}, \qquad y = 3 + (4-t)^{3/2}, \qquad 0 \leq t \leq 4$$

$$\begin{aligned} L &= \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^4 \sqrt{\left(\frac{1}{2}t^{1/2}\right)^2 + \left(-\frac{3}{2}(4-t)^{1/2}\right)^2}\, dt \\ &= \int_0^4 \sqrt{\frac{1}{4}t + \frac{9}{4}(4-t)}\, dt = \int_0^4 \sqrt{\frac{1}{4}(t + 36 - 9t)}\, dt = \int_0^4 \frac{1}{2} \cdot \sqrt{36 - 8t}\, dt \\ &= \frac{1}{2}\int_{36}^{4} \sqrt{u} \cdot \left(-\frac{1}{8}du\right) = -\frac{1}{16}\int_{36}^{4} u^{1/2}\, du = -\frac{1}{16}\left[\frac{2}{3}u^{3/2}\right]_{36}^{4} \\ &= -\frac{1}{24}\left[4^{3/2} - 36^{3/2}\right] = -\frac{1}{24}\left[(2^2)^{3/2} - (2^2 \times 3^2)^{3/2}\right] = -\frac{1}{24}\left[2^3 - (2 \times 3)^3\right] \\ &= \frac{1}{2}\left[6^3 - 2^3\right]. \end{aligned}$$