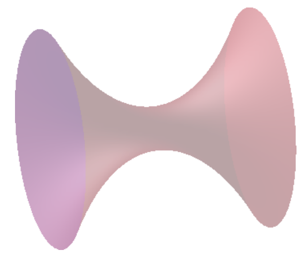
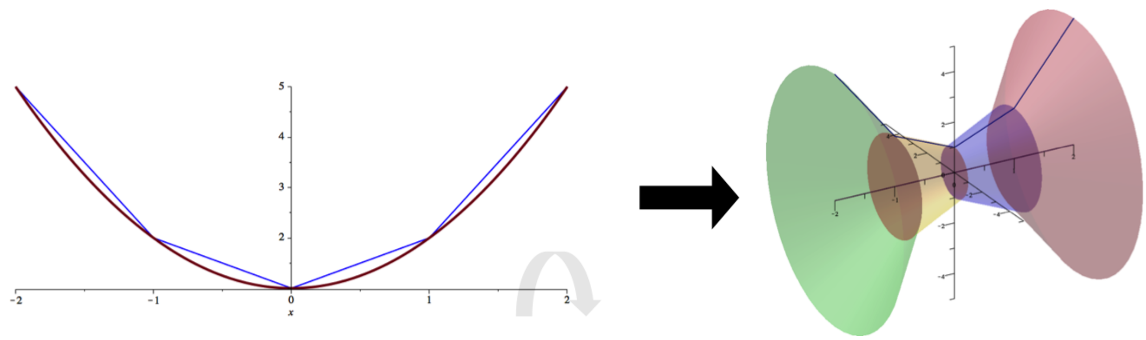
Suppose you are designing a part for a system. For convenience, you assume that the shape of this part will be given by rotating a curve, about the x-axis, as shown in the figure.
You wish to calculate the amount of material required to cover the part, that is you need to calculate its surface area.

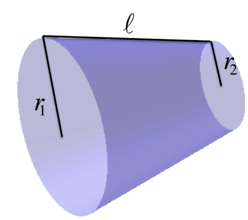


Surface area of a solid of revolution

We start by approximating the curve using linear segments, and then rotating each segment, as shown in the figure below



For each of the sections, known as a *frustum*, we can calculate the surface area as
$A_s = 2\pi r\ell = 2\pi\left(\frac{r_1+r_2}{2}\right)\ell = \pi\left(r_1+r_2\right)\ell,$



in our approximation, the frustum i will have

$$r_1 = f(x_{i-1}) \qquad r_2 = f(x_i), \qquad \ell = |P_{i-1}-P_i| = \sqrt{\Delta x^2+\Delta y^2}.$$

The surface area of the solid will be given by the sum of the surface areas of each frustum,

$$A_s = \sum_{i=1}^{n}\pi\left[f(x_{i-1})+f(x_i)\right]\sqrt{\Delta x^2+\Delta y^2} = \sum_{i=1}^{n}\pi\left[f(x_{i-1})+f(x_i)\right]\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x,$$

If we take the limit when each section is infinitesimally small ($\Delta x \to 0$ or equivalently $n\to\infty$), we obtain

$$\begin{aligned} A_s &= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\pi\left[f(x_{i-1})+f(x_i)\right]\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x\right] \\ &= \int 2\pi f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx. \end{aligned}$$

Where we used the fact that

$$\lim_{\Delta x\to 0} f(x_{i-1}) = \lim_{\Delta x\to 0} f(x_i) = f(x)$$

In general,

$$\begin{aligned} A_s &= \int 2\pi y ds && \text{rotation about } x-\text{axis} \\ A_s &= \int 2\pi x ds && \text{rotation about } y-\text{axis} \end{aligned}$$

where,

$$\begin{aligned} ds &= \sqrt{1+\left(\tfrac{dy}{dx}\right)^2}dx, && \text{if } y=f(x),\quad a\le x\le b \\ ds &= \sqrt{1+\left(\tfrac{dx}{dy}\right)^2}dy, && \text{if } x=g(y),\quad c\le y\le d \end{aligned}$$

Example: Determine the surface area of the solid obtained by rotating $y=\sqrt[3]{x}$, for $0\le y\le 1$ about the y-axis.

We first find ds as

$$\frac{dy}{dx}=\frac{1}{3}x^{-\frac{2}{3}}, \qquad ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{1}{9x^{\frac{4}{3}}}}=\sqrt{\frac{9x^{\frac{4}{3}}+1}{9x^{\frac{4}{3}}}}=\frac{\sqrt{9x^{\frac{4}{3}}+1}}{3x^{\frac{2}{3}}}.$$

Next we need to find the limits of the integral, when $y=0$, $x=0$ and when $y=1$, $x=1$, so that our surface area is found as

$$A_s=\int_0^1 2\pi\, x\,\frac{\sqrt{9x^{\frac{4}{3}}+1}}{3x^{\frac{2}{3}}}dx=\frac{2}{3}\pi\int_0^1 x^{\frac{1}{3}}\sqrt{9x^{\frac{4}{3}}+1}\,dx.$$

To solve this integral we use the substitution

$$u=9x^{\frac{4}{3}}+1, \qquad du=12x^{\frac{1}{3}}dx,$$

and the integral becomes

$$\begin{aligned} A_s &= \frac{\pi}{18}\int_1^{10}\sqrt{u}\,du \\ &= \frac{\pi}{27}\,u^{\frac{3}{2}}\Big|_1^{10} \\ &= \frac{\pi}{27}\left(10^{\frac{3}{2}}-1\right)\approx 1.134\pi \end{aligned}$$