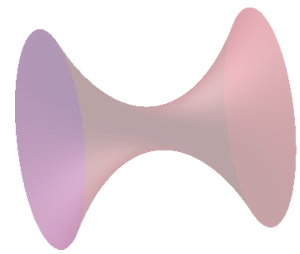
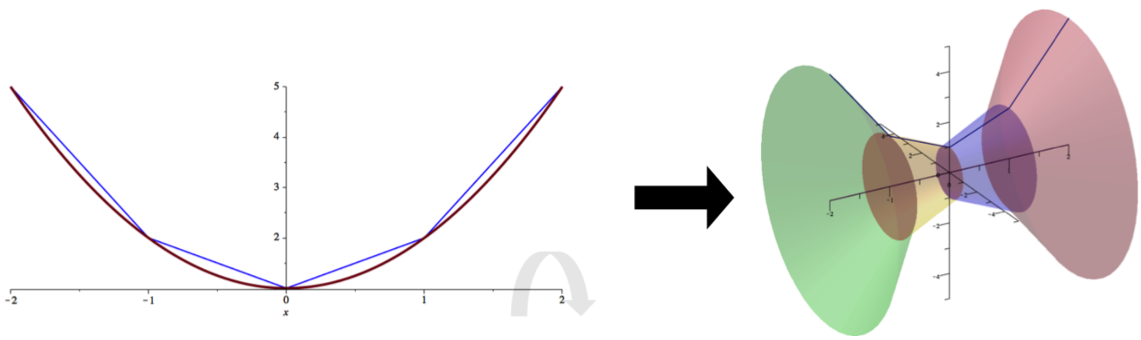
Suppose you are designing a part for a system. For convenience, you assume that the shape of this part will be given by rotating a curve, about the x-axis, as shown in the figure.
You wish to calculate the amount of material required to cover the part, that is you need to calculate its surface area.

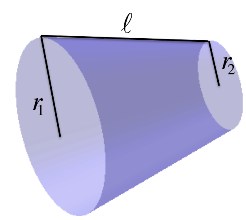


Surface area of a solid of revolution

We start by approximating the curve using linear segments, and then rotating each segment, as shown in the figure below



For each of the sections, known as a *frustum*, we can calculate the surface area as
$A_s = 2\pi r\ell = 2\pi\left(\frac{r_1+r_2}{2}\right)\ell = \pi\left(r_1+r_2\right)\ell,$



in our approximation, the frustum i will have

$$r_1 = f(x_{i-1}) \qquad r_2 = f(x_i), \qquad \ell = |P_{i-1} - P_i| = \sqrt{\Delta x^2 + \Delta y^2}.$$

The surface area of the solid will be given by the sum of the surface areas of each frustum,

$$A_s = \sum_{i=1}^{n} \pi\left[f(x_{i-1}) + f(x_i)\right]\sqrt{\Delta x^2 + \Delta y^2} = \sum_{i=1}^{n} \pi\left[f(x_{i-1}) + f(x_i)\right]\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x,$$

If we take the limit when each section is infinitesimally small ($\Delta x \to 0$ or equivalently $n \to \infty$), we obtain

$$\begin{aligned} A_s &= \lim_{n\to\infty}\left[\sum_{i=1}^{n} \pi\left[f(x_{i-1}) + f(x_i)\right]\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x\right] \\ &= \int 2\pi f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx. \end{aligned}$$

Where we used the fact that

$$\lim_{\Delta x\to 0} f(x_{i-1}) = \lim_{\Delta x\to 0} f(x_i) = f(x)$$

In general,

$$A_s = \int 2\pi y ds \qquad \text{rotation about } x-\text{axis}$$
$$A_s = \int 2\pi x ds \qquad \text{rotation about } y-\text{axis}$$

where,

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx, \qquad \text{if } y = f(x), \quad a \leq x \leq b$$
$$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}dy, \qquad \text{if } x = g(y), \quad c \leq y \leq d$$

Exercise: Determine the surface area of the solid obtained by rotating $y = \sqrt[3]{x}$, for $0 \leq y \leq 1$ about the y-axis.