Proper vs. improper rational functions

Integration by partial fractions works only with *proper rational expressions*, but not with *improper rational expressions*. A fraction a/b is proper if the numerator (disregarding sign) is less than the denominator, and improper otherwise. The same is true for rational expressions, but instead of comparing the value of the numerator and denominator, you compare their *degrees*. The degree of a polynomial is its **highest** power of x.

Expression	Degree of numerator	Degree of denominator	Type
$\dfrac{x^2+5}{x^3}$	2	3	proper
$\dfrac{x^2+5}{x}$	2	1	improper
$\dfrac{x^4}{x^3+1}$	4	3	improper
$\dfrac{x^2-x+3}{x^5+x^3-2x-1}$	2	5	proper

If the rational expression is improper we use long division to simplify it; if it is proper we use partial fractions.

Polynomial factorization

- Greatest common factor

$$8x^4-4x^3+10x^2 = 2x^2\left(4x^2-2x+5\right)$$

- Factoring by grouping

$$x^5-3x^3-2x^2+6 = \left(x^5-3x^3\right)-\left(2x^2-6\right) = x^3\left(x^2-3\right)-2\left(x^2-3\right) = \left(x^3-2\right)\left(x^2-3\right)$$

- Special forms quadratic polynomials

$$\begin{aligned} a^2+2\,a\,b+b^2 &= (a+b)^2 \\ a^2-2\,a\,b+b^2 &= (a-b)^2 \\ a^2-b^2 &= (a+b)(a-b) \end{aligned}$$

- Special forms cubic polynomials

$$\begin{aligned} a^3+b^3 &= (a+b)(a^2-ab+b^2) \\ a^3-b^3 &= (a-b)(a^2+ab+b^2) \end{aligned}$$

- Higher degree polynomials

$$\begin{aligned} x^4-6x^2+9 &= \left(x^2\right)^2-6\left(x^2\right)+9 \\ &= u^2-6u+9 \\ &= (u-3)^2 \\ &= \left(x^2-3\right)^2 \end{aligned}$$

Polynomial factorization, cont.

- Factoring quadratic polynomials using the quadratic formula

$$ax^2 + bx + c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example : $2x^2 - 5x - 3$

$$\begin{aligned} x = \frac{-(-5) \pm \sqrt{5^2 - 4 \cdot 2 \cdot (-3)}}{2 \cdot 2} &= \frac{5 \pm \sqrt{25 + 24}}{4} \\ &= \frac{5 \pm \sqrt{49}}{4} = \frac{5 \pm 7}{4} \end{aligned}$$

$$x_1 = \frac{12}{4} = 3 \qquad x_2 = \frac{-2}{4} = -\frac{1}{2}$$

The factorization becomes $(x - x_1)(x - x_2) = (x - 3)\left(x + \frac{1}{2}\right)$

Partial fractions with linear factors in the denominator

Distribute into partial fractions with constants in the numerators.

Example

Find the integral: $\displaystyle\int \frac{x-1}{x^2+x}\,dx.$

Partial fractions

$$\frac{x-1}{x^2+x} = \frac{x-1}{x\,(x+1)} = \frac{A}{x} + \frac{B}{x+1}.$$

Multiply both sides by $x\,(x+1)$

$$\begin{aligned} \frac{x-1}{x\,(x+1)} \cdot [x\,(x+1)] &= \frac{A}{x} \cdot [x\,(x+1)] + \frac{B}{x+1} \cdot [x\,(x+1)] \\ x - 1 &= A(x+1) + B\,x \end{aligned}$$

Equate coefficients

- Coefficients of $x^1 = x$ $\qquad 1 = A + B$
- Coefficients of $x^0 =$ constants $\qquad -1 = A$

Solve for A and B $\qquad A = -1, \qquad B = 1 - A = 1 - (-1) = 1 + 1 = 2$

The resulting partial fraction is $\displaystyle\frac{x-1}{x^2+x} = \frac{-1}{x} + \frac{2}{x+1}.$

And to solve the integral

$$\begin{aligned} \int \frac{x-1}{x^2+x}\,dx &= -\int \frac{1}{x}\,dx + 2\int \frac{1}{x+1}\,dx \\ &= \boxed{-\ln|x| + 2\ln|x+1| + C.} \end{aligned}$$

Partial fractions with quadratic factors in the denominator

Use linear equation in the numerator of the quadratic factor.

Example

Find the integral: $\displaystyle\int \frac{2x-1}{x^3+x}\,dx.$

Partial fractions

$$\frac{2x-1}{x^3+x} = \frac{2x-1}{x\,(x^2+1)} \quad = \quad \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Multiply both sides by $x\,(x^2+1)$

$$\begin{aligned}
\frac{2x-1}{x\,(x^2+1)} \cdot \left[x\,(x^2+1)\right] &= \frac{A}{x} \cdot \left[x\,(x^2+1)\right] + \frac{Bx+C}{x^2+1} \cdot \left[x\,(x^2+1)\right] \\
2x-1 &= A(x^2+1) + (Bx+C)\,x \\
2x-1 &= Ax^2 + A + Bx^2 + Cx
\end{aligned}$$

Equate coefficients

- Coefficients of x^2 $\qquad 0 = A + B$
- Coefficients of x $\qquad 2 = C$
- Coefficients of x^0 = constants $\qquad -1 = A$

Solve for A, B and C $\qquad A = -1, \qquad B = -A = -(-1) = 1 \qquad C = 2$

The resulting partial fraction is $\displaystyle\frac{2x-1}{x^3+x} = \frac{-1}{x} + \frac{x+2}{x^2+1}.$

And the integral becomes

$$\begin{aligned}
\int \frac{2x-1}{x^3+x}\,dx &= -\int \frac{1}{x}\,dx + \int \frac{x+2}{x^2+1}\,dx \\
&= -\int \frac{1}{x}\,dx + \int \frac{x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}\,dx
\end{aligned}$$

To solve the second integral on the right hand side (RHS) we use the substitution $u = x^2+1$, $du = 2x\,dx$

$$\int \frac{2x-1}{x^3+x}\,dx = -\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{u}\,du + 2\int \frac{1}{x^2+1}\,dx$$

To solve the last integral on the RHS we use the substitution $x = \tan v$, $dx = \sec^2 v\,dv$

$$\begin{aligned}
\int \frac{2x-1}{x^3+x}\,dx &= -\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{u}\,du + 2\int \frac{1}{\tan^2 v+1}\,(\sec^2 v\,dv) \\
&= -\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{u}\,du + 2\int \frac{\sec^2 v}{\sec^2 v}dv \\
&= -\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{u}\,du + 2\int dv \\
&= -\ln|x| + \frac{1}{2}\ln|u| + 2v + C \\
&= \boxed{-\ln|x| + \frac{1}{2}\ln\left|x^2+1\right| + 2\tan^{-1} x + C.}
\end{aligned}$$

Partial fractions with repeated factors in the denominator

Add a term for every repetition.

Example

Find the integral: $\displaystyle\int \frac{2x^3+4x-1}{(x+1)^2(x^2+1)}\,dx.$

Partial fractions

$$\frac{2x^3+4x-1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+1)}$$

Multiply both sides by $(x+1)^2(x^2+1)$

$$\frac{2x^3+4x-1}{(x+1)^2(x^2+1)} \cdot \left[(x+1)^2(x^2+1)\right] = \frac{A}{x+1}\cdot\left[(x+1)^2(x^2+1)\right] + \frac{B}{(x+1)^2}\cdot\left[(x+1)^2(x^2+1)\right] + \frac{Cx+D}{x^2+1}\cdot\left[(x+1)^2(x^2+1)\right]$$

$$\begin{aligned}
2x^3+4x-1 &= A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2 \\
&= A(x^3+x^2+x+1) + B(x^2+1) + (Cx+D)(x^2+2x+1) \\
&= Ax^3+Ax^2+Ax+A+Bx^2+B+Cx^3+2Cx^2+Cx+Dx^2+2Dx+D
\end{aligned}$$

Equate coefficients

- Coefficients of x^3 $\boxed{2 = A + C}$

$$\boxed{2x^3} + 4x - 1 = \boxed{Ax^3} + Ax^2 + Ax + A + Bx^2 + B + \boxed{Cx^3} + 2Cx^2 + Cx + Dx^2 + 2Dx + D$$

- Coefficients of x^2 $\boxed{0 = A + B + 2C + D}$

$$2x^3 + 4x - 1 = Ax^3 + \boxed{Ax^2} + Ax + A + \boxed{Bx^2} + B + Cx^3 + \boxed{2Cx^2} + Cx + \boxed{Dx^2} + 2Dx + D$$

- Coefficients of x^1 $\boxed{4 = A + C + 2D}$

$$2x^3 + \boxed{4x} - 1 = Ax^3 + Ax^2 + \boxed{Ax} + A + Bx^2 + B + Cx^3 + 2Cx^2 + \boxed{Cx} + Dx^2 + \boxed{2Dx} + D$$

- Coefficients of x^0 $\boxed{-1 = A + B + D}$

$$2x^3 + 4x\boxed{-1} = Ax^3 + Ax^2 + Ax + \boxed{A} + Bx^2 + \boxed{B} + Cx^3 + 2Cx^2 + Cx + Dx^2 + 2Dx + \boxed{D}$$

Partial fractions with repeated factors in the denominator, cont.

Continuation example

Solve the equations

$$\begin{aligned} 2 &= A + C && (1)\\ 0 &= A + B + 2C + D && (2)\\ 4 &= A + C + 2D && (3)\\ -1 &= A + B + D && (4)\\ & && (5) \end{aligned}$$

- From Eqn. (1) $\boxed{C = 2 - A}$
- From Eqn. (2)

$$\begin{aligned} 0 &= A + B + 2C \qquad + D \\ &= A + B + 2(2 - A) + D = A + B + 4 - 2A + D = -A + B + D + 4 \end{aligned}$$

  and $B = A - D - 4$.
- From Eqn. (3)

$$\begin{aligned} 4 &= A + C \qquad + 2D \\ &= A + (2 - A) + 2D = 2 + 2D \end{aligned}$$

  and $\boxed{D = 1}$ and $B = A - D - 4 = A - 1 - 4$, so $\boxed{B = A - 5}$.
- From Eqn. (4)

$$\begin{aligned} -1 &= A + B + D \\ &= A + (A - 5) + 1 = 2A - 5 + 1 = 2A - 4 \end{aligned}$$

  and $\boxed{A = \frac{3}{2}}$.

So that $A = \frac{3}{2}$, $B = A - 5 = -\frac{7}{2}$, $C = 2 - A = \frac{1}{2}$ and $D = 1$

The resulting partial fraction is $\dfrac{2x^3 + 4x - 1}{(x+1)^2(x^2+1)} = \dfrac{3}{2}\dfrac{1}{x+1} - \dfrac{7}{2}\dfrac{1}{(x+1)^2} + \dfrac{\frac{1}{2}x + 1}{x^2+1}$.

To solve the integral

$$\begin{aligned} \int \frac{2x^3 + 4x - 1}{(x+1)^2(x^2+1)}\,dx &= \frac{3}{2}\int \frac{dx}{x+1} - \frac{7}{2}\int \frac{dx}{(x+1)^2} + \int \frac{\frac{1}{2}x + 1}{x^2+1}\,dx \\ &= \frac{3}{2}\int \frac{dx}{x+1} - \frac{7}{2}\int \frac{dx}{(x+1)^2} + \frac{1}{2}\int \frac{x\,dx}{x^2+1} + \int \frac{dx}{x^2+1} \\ &= \boxed{\frac{3}{2}\ln|x+1| + \frac{7}{2(x+1)} + \frac{1}{4}\ln\left|x^2+1\right| + \tan^{-1}x + C.} \end{aligned}$$