Basic integration

In its most basic form, using the Fundamental Theorem of Calculus, an indefinite integral is simply

$$\int f(x)\ dx = F(x) + C,$$

where C is an arbitrary constant and $F(x)$ is the antiderivative of $f(x)$, that is $F'(x) = f(x)$.

If the integral is definite,

$$\int_{a_1}^{a_2} f(x)\ dx = F(a_2) - F(a_1).$$

Table of integrals

Using the derivatives that we already know from Calculus I, we can calculate several integrals like the ones summarized in the table below.

1. $\displaystyle\int a\ dx = ax + C$

2. $\displaystyle\int (x+a)^n\ dx = \frac{1}{n+1}(x+a)^{n+1}, \quad n \neq -1 + C$

3. $\displaystyle\int e^{ax}\ dx = \frac{1}{a}e^{ax} + C$

4. $\displaystyle\int \frac{1}{ax+b}\ dx = \frac{1}{a}\ln|ax+b| + C$

5. $\displaystyle\int a^x\ dx = \frac{1}{\ln|a|} + C, \qquad (a > 0, a \neq 1)$

6. $\displaystyle\int \sin ax\ dx = -\frac{1}{a}\cos ax + C$

7. $\displaystyle\int \cos ax\ dx = \frac{1}{a}\sin ax + C$

8. $\displaystyle\int \sec^2 ax\ dx = \frac{1}{a}\tan ax + C$

9. $\displaystyle\int \csc^2 ax\ dx = -\frac{1}{a}\cot ax + C$

10. $\displaystyle\int \sec ax \tan ax\ dx = \frac{1}{a}\sec ax + C$

11. $\displaystyle\int \csc ax \cot ax\ dx = -\frac{1}{a}\csc ax + C$

12. $\displaystyle\int \tan ax\ dx = \frac{1}{a}\ln|\sec ax| + C$

13. $\displaystyle\int \cot ax\ dx = \frac{1}{a}\ln|\sin ax| + C$

14. $\displaystyle\int \sec ax\ dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$

15. $\displaystyle\int \csc ax\ dx = -\frac{1}{a}\ln|\csc ax + \cot ax| + C$

16. $\displaystyle\int \sinh ax\ dx = \frac{1}{a}\cosh x + C$

17. $\displaystyle\int \cosh ax\ dx = \frac{1}{a}\sinh x + C$

18. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} x\left(\frac{x}{a}\right) + C$

19. $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

20. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\ \frac{x}{a}\ + C$

21. $\displaystyle\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C, \qquad (a > 0)$

22. $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C, \qquad (x > a > 0)$

Integration by substitution

The basic idea is to take an integral of the form $\int f(x)\,dx$, for which we don't know the antiderivative, and transform it to an integral of the form

$$\int g(u)\,du,$$

where we do know the antiderivative of g, perhaps one of the functions in the table above.

$u\,du$ form

The easiest case is when we can obtain a transformation of the form

$$\int f(x)\,dx = \int u\,du.$$

Example

Find the integral: $\int \left(x+\sqrt{3}\right)dx.$

If we make the substitution $u = x+\sqrt{3},$
differentiating both sides gives $du = dx,$

making this substitution on the integrand gives

$$\int \left(x+\sqrt{3}\right)dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\left(x+\sqrt{3}\right)^2 + C.$$

Integration by substituting $u \quad ax+b$

We can easily add an exponent to the previous example and the same principle applies

$$\int \left(x+\sqrt{3}\right)^3 dx,$$

using the same substitution, $u = x+\sqrt{3}$, we obtain

$$\int \left(x+\sqrt{3}\right)^3 dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}\left(x+\sqrt{3}\right)^4 + C.$$

In general, whenever the integral is of the form $\int f\,(ax+b)\,dx$, we can use this substitution technique.

Example

Find the integral: $\int \sin(2x+3)\,dx.$

Using the substitution, $u = 2x+3$, gives $du = 2dx$ or $dx = \frac{du}{2}$

$$\begin{aligned}\int \sin(2x+3)\,dx = \int \sin(u)\left(\frac{1}{2}du\right) &= \frac{1}{2}\int \sin u\,du \\ &= -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(2x+3) + C\end{aligned}$$

Finding $\int f(g(x))g'(x)\,dx$ by substituting u $g(x)$

The key is to recognize that the derivative of one part of the function is multiplying the rest of the function.

To evaluate

$$\int f\left(g(x)\right) g'(x)\,dx$$

substitute $u = g(x)$, and $du = g'(x)\,dx$ to give

$$\int f(u)\,du$$

Example

Find the integral: $\displaystyle\int 2x\left(x^2+\sqrt{3}\right)^3 dx.$

If we consider $u = x^2 + \sqrt{3}$, we obtain $du = 2x\,dx$ so that

$$\begin{aligned}\int 2x\left(x^2+\sqrt{3}\right)^3 dx &= \int u^3\,du\\ &= \frac{1}{4}u^4 + C = \frac{1}{4}\left(x^2+\sqrt{3}\right)^4 + C.\end{aligned}$$

Complete the square

Several of the known integrals in the table above include a perfect square, $\left(x^2 \pm a^2\right)$, so that we can transform a given integral into one of those forms by completing the square.

Recall

$$(x \pm b)^2 = x^2 \pm 2\cdot x\cdot b + b^2.$$

Example

Find the integral: $\displaystyle\int \frac{8}{x^2-2x+2}\,dx.$

A perfect square including the terms $x^2 - 2x$ should have $+1$ as the last term. To obtain this we separate the last term as follows

$$x^2 - 2x + 2 = x^2 - 2x + 1 + 1 = (x-1)^2 + 1,$$

giving

$$\int \frac{8}{x^2-2x+2}\,dx = 8\int \frac{1}{(x-1)^2+1}\,dx.$$

Now if we make the substitution $u = x - 1$ and $du = dx$ and the resulting integral is

$$\begin{aligned}8\int \frac{1}{(x-1)^2+1}\,dx &= \int \frac{1}{u^2+1}\,du\\ &= 8\tan^{-1}u + C = 8\tan^{-1}(x-1) + C.\end{aligned}$$

Improper fractions

Recall that an improper fraction is characterized by a greater degree in the numerator than in the denominator. In this case we perform long division to simplify our integrand.

Example

Find the integral: $$\int \frac{4x^3 - x^2 + 16x}{x^2+4} dx.$$

Performing the long division gives,

$$\begin{array}{r|l} & \quad 4x - 1 \\ \hline x^2+4 & 4x^3 - x^2 + 16x \\ & -4x^3 \qquad - 16x \\ \hline & \qquad -x^2 + 0x \\ & \qquad\ \ x^2 \quad\ + 4 \\ \hline & \qquad\qquad 0x + 4 \end{array}$$

so that

$$\frac{4x^3 - x^2 + 16x}{x^2+4} = 4x - 1 + \frac{4}{x^2+4}.$$

Our integral becomes,

$$\int \frac{4x^3 - x^2 + 16x}{x^2+4} dx = \int \left[4x - 1 + \frac{4}{x^2+4}\right] dx = \int 4x\, dx - \int dx + \int \frac{4}{x^2+4} dx,$$

all of these are part of our *basic functions* table and we can readily find the solution as

$$\int \frac{4x^3 - x^2 + 16x}{x^2+4} dx = 2x - x + 2\tan^{-1}\left(\frac{x}{2}\right) + C.$$

Solving for x in the u definition

Example

Consider the following integral

$$\int \frac{1}{\left(1+\sqrt{x}\right)^3} dx,$$

If we consider $u = 1 + \sqrt{x}$, we obtain $du = \frac{1}{2\sqrt{x}} dx$. Solving for dx gives $2\sqrt{x} du = dx$, if we solve for x in the definition of u we obtain $\sqrt{x} = u - 1$, which gives $2\,(u-1)\,du = dx$, given the following integral

$$\int \frac{1}{\left(1+\sqrt{x}\right)^3} dx = \int \frac{2\,(u-1)}{u^3} du,$$

and finally we can solve our integral as

$$\begin{aligned} \int \frac{1}{\left(1+\sqrt{x}\right)^3} dx = \int \frac{2\,(u-1)}{u^3} du &= 2\int u^{-2} du \ -2\int u^{-3} du \\ &= -2u^{-1} \qquad +u^{-2} \qquad + C = -\frac{2}{1+\sqrt{x}} + \frac{1}{\left(1+\sqrt{x}\right)^2} + C. \end{aligned}$$

Using $(a+b)(a-b)$ (a^2-b^2)

Example

Consider the following integral $$\int \sqrt{\frac{1+x}{1-x}}dx,$$

we can perform the following calculation to simplify the integrand

$$\frac{\sqrt{1+x}}{\sqrt{1-x}} = \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}} = \frac{1+x}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}.$$

The simplified integral is

$$\int \sqrt{\frac{1+x}{1-x}}dx = \int \left[\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right] dx = \int \frac{1}{\sqrt{1-x^2}}dx + \int \frac{x}{\sqrt{1-x^2}}dx$$

The first integral is formula 18 in the table above and for the second we use the substitution $u = 1-x^2$, $du = -2x\,dx$, giving

$$\begin{aligned} \int \sqrt{\frac{1+x}{1-x}}dx &= \int \frac{1}{\sqrt{1-x^2}}dx + \int \frac{x}{\sqrt{1-x^2}}dx \\ &= \sin^{-1} x \quad -\sqrt{1-x^2} \quad + C. \end{aligned}$$

Manipulating exponentials

Example

Consider the following integral

$$\int \frac{1}{\sqrt{e^{2x}-1}}dx.$$

Multiplying the integrand by $\frac{e^x}{e^x}$ we obtain,

$$\int \frac{1}{\sqrt{e^{2x}-1}}dx = \int \frac{e^x}{e^x\sqrt{e^{2x}-1}}dx,$$

using the substitution $u = e^x$, with $du = e^x dx$ we obtain

$$\begin{aligned} \int \frac{e^x}{e^x\sqrt{e^{2x}-1}}dx &= \int \frac{1}{u\sqrt{u^2-1}}du \\ &= \sec^{-1}|u| + C = \sec^{-1}(e^x) + C. \end{aligned}$$

Summary

There are occasions when it is possible to perform an apparently difficult integral by using a substitution. The substitution changes the variable and the integrand, and when dealing with definite integrals, the limits of integration can also change.

The only way to master the techniques explained here is by working on a LOT of practice exercises until they become second nature.