Problem: *The time-dependent temperature of an object changes at a rate proportional to the difference between the temperature of its surroundings and the temperature of the object. This relation is expressed as the Newton's Law of cooling and is written as:*

$$\rho c \frac{dT(t)}{dt} = -\frac{hA_s}{V}\left(T(t) - T_s(t)\right),$$

where ρ is the density of the object, c is its specific heat, h is the heat transfer coefficient between the object and its surroundings, A_s is the surface area of the object, V is the volume of the object, and $T_s(t)$ is the temperature of the surroundings.
For simplicity, the equation can be expressed as,

$$\frac{dT(t)}{dt} + mT(r) = mT_s(t), \qquad m = \frac{hA_s}{\rho c V}.$$

The solution of this equation can be found through the use of an integrating factor as,

$$T(t) = me^{-mt}\int T_s(t)e^{mt}dt.$$

Assume that the temperature of the surrounding medium is linearly changing in time as,

$$T_s(t) = \alpha t + T_0,$$

where T_0 is the initial temperature of the surroundings and α is the rate at which T_s is changing. The expression for the temperature is then given by,

$$T(t) = me^{-mt}\int \left(\alpha t + T_0\right)e^{mt}dt. \tag{1}$$

Solution:

To solve Eqn. (1) we need to integrate by parts with

$$u = \left(\alpha t + T_0\right), \qquad dv = e^{mt}dt$$
$$du = \alpha\, dt, \qquad v = \frac{1}{m}e^{mt}$$

giving

$$\begin{aligned} T(t) &= me^{-mt}\int \left(\alpha t + T_0\right)e^{mt}dt = me^{-mt}\left[\left(\alpha t + T_0\right)\left(\frac{1}{m}e^{mt}\right) - \int\left(\frac{1}{m}e^{mt}\right)\alpha dt\right] \\ &= me^{-mt}\left[\frac{1}{m}\left(\alpha t + T_0\right)e^{mt} - \frac{\alpha}{m}\int e^{mt}dt\right] = me^{-mt}\left[\frac{1}{m}\left(\alpha t + T_0\right)e^{mt} - \frac{\alpha}{m^2}e^{mt} + C\right] \\ &= \alpha t + T_0 - \frac{\alpha}{m} + Cme^{-mt}. \end{aligned}$$

Problem 1: Assume that a thermometer with a spherical test section of radius $r = 0.3$cm is initially at a temperature T_i=50^{o}C, the density of the thermometer medium is ρ =1000kg/m^3, its specific heat is c=3000J/kg^{o}C, the heat transfer coefficient h=1500W/m^{2o}C, the initial temperature of the surroundings T_0=100^{o}C, and the rate of change of the surroundings temperature $\alpha = 5^o$C/s. How does the temperature of the thermometer compares to that of the surrounding medium at long times? Plot the ratio of the thermometer's temperature to the surrounding temperature as a function of time.

We use the initial temperature of the thermometer T_i to find C,

$$\begin{aligned} T(0) = T_i &= \alpha(0) + T_0 - \frac{\alpha}{m} + Cme^0 = T_0 - \frac{\alpha}{m} + mC \\ &\Rightarrow \\ C &= \frac{1}{m}\left(T_i - T_0 + \frac{\alpha}{m}\right) \end{aligned}$$

$$\frac{T(t)}{T_s(t)} = \frac{\left(\alpha t + T_0 - \frac{\alpha}{m}\right) + \left(T_i - T_0 + \frac{\alpha}{m}\right)e^{-mt}}{\alpha t + T_0}.$$

Note that we can write the previous relation as

$$\frac{T(t)}{T_s(t)} = 1 + \frac{\left(T_i - T_0 + \frac{\alpha}{m}\right)e^{-mt}}{\alpha t + T_0} - \frac{\frac{\alpha}{m}}{\alpha t + T_0},$$

since

$$\lim_{t\to\infty} \frac{e^{-mt}}{\alpha t + T_0} = 0, \qquad \text{and} \qquad \lim_{t\to\infty} \frac{1}{\alpha t + T_0} = 0$$

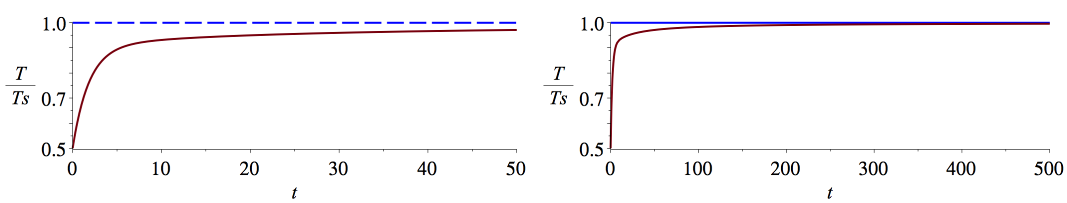
we get

$$\frac{T(t)}{T_s(t)} \to 1, \quad \text{as} \quad t \to \infty,$$

that is, at very long times the temperature of the thermometer and that of the surrounding medium are equal to each other. However, at small times there is a lag between the two temperatures as shown in the figure below.

To calculate the figure we first need to find the value of the different model parameters:

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(3\times10^{-3}\right)^3 \approx 1.131\times10^{-7}\text{m}^3 \\ A_s &= 4\pi r^2 = 4\pi\left(3\times10^{-3}\right)^2 \approx 1.131\times10^{-4}\text{m}^2 \\ m &= \frac{1500\times1.131\times10^{-4}}{1000\times3000\times1.131\times10^{-7}} = 0.5 \end{aligned}$$



And as expected at long times, the thermometer's temperature converges towards that of the surrounding medium.

Problem 2: Assume that for the problem above the surrounding temperature is changing periodically as $T_s(t) = \cos(2\pi t)$. Find an expression for the temperature of the thermometer as a function of time.

The integral becomes

$$T(t) = me^{-mt} \int \cos(2\pi t)\, e^{mt} dt.$$

We use integration by parts

$$\begin{aligned} T(t) = me^{-mt} \int \cos(2\pi t)\, e^{mt} dt &= me^{-mt}\left[\frac{me^{mt}\cos(2\pi t)}{4\pi^2+m^2} + \frac{2\pi e^{mt}\sin(2\pi t)}{4\pi^2+m^2}\right] + C \\ &= \frac{m^2\cos(2\pi t)}{4\pi^2+m^2} + \frac{2m\pi\sin(2\pi t)}{4\pi^2+m^2} + C \end{aligned}$$

To find C we use the initial condition, $T(0) = 50$,

$$\begin{aligned} T(0) = 50 &= \frac{m^2\cos(0)}{4\pi^2+m^2} + \frac{2m\pi\sin(0)}{4\pi^2+m^2} + C \\ &\Rightarrow \quad C = 50 - \frac{m^2}{4\pi^2+m^2} \end{aligned}$$

The solution is then,

$$T(t) = \frac{m^2\cos(2\pi t)}{4\pi^2+m^2} + \frac{2m\pi\sin(2\pi t)}{4\pi^2+m^2} + 50 - \frac{m^2}{4\pi^2+m^2}$$