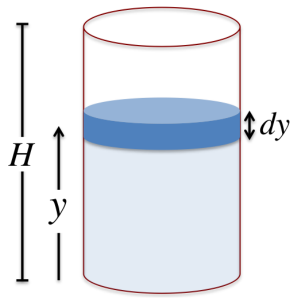
1 Finding work required to pump liquid from a tank

Problem: *Find the work done by pumping out water from the top of a cylindrical tank 1 m in radius and 4 m tall, if the tank is initially full. (The density of water is 1000 kg/m^3).*



Pumping liquid out of the top of a tank requires work because the liquid is moving against gravity. To calculate this, we imagine the work required to lift small disks of liquid up and out of the tank. We are asked to calculate the work performed in all of this activity. Recall that work (W) is defined as force (F) times distance (d),

$$W = F \cdot d.$$

In this example the force is given by the weight = mass times gravity ($F_W = m \cdot g$), where F_W is weight, m is mass, and g is the gravitational constant, 9.8 m/s^2. For each disk,

$$F_W = [\text{density}] \cdot [\text{volume}] \cdot [\text{gravity}] = \rho \cdot \left[\pi r^2 dy\right] \cdot [g] = \pi \rho r^2 g dy.$$

The distance each disk has to be lifted depends on its position with respect to the top of the tank: $d = H - y$.

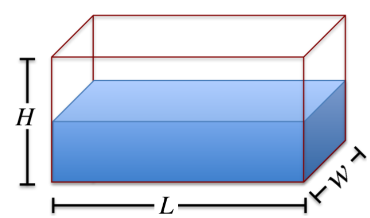
The total work would be the sum of the work done in the individual disks, so that

$$W = \int_0^H \left[\pi \rho r^2 g dy\right] \cdot [H - y] = \int_0^H \pi \rho r^2 g\,(H - y)\,dy,$$

since the density and the radius are constants, the resulting integral is

$$\begin{aligned} W &= \pi \rho r^2 g \int_0^H (H - y)\,dy \\ &= \pi \rho r^2 g \left[Hy - \frac{y^2}{2}\Big|_0^H\right] = \pi \rho r^2 g \left[H(H-0) - \frac{1}{2}(H^2 - 0)\right] = \pi \rho r^2 g \left[H^2 - \frac{1}{2}H^2\right] = \frac{\pi \rho r^2 g H^2}{2} \\ &= \frac{3.1416 \times 1000 \times 1^2 \times 9.8 \times 4^2}{2} \approx 246,176 \text{ N m}. \end{aligned}$$

Exercise: A rectangular tank ($H \times L \times w = 8$ m $\times$ 10 m $\times$ 4 m) contains 160 m^3 of water. How much work is needed to pump all of the water out of the tank? How would your integral change if the tank is on one of its other sides? Which position would produce the greatest amount of work?



- If the tank is on its length (as in the figure), the total work done is

$$\begin{aligned} W = \int_0^{H/2} [\rho L w g dy] \cdot [H - y] &= \rho L w g \int_0^{H/2} (H - y) dy \\ &= \rho L w g \left[Hy - \frac{y^2}{2}\Big|_0^{H/2}\right] = \rho L w g \left[\frac{H^2}{2} - \frac{H^2}{8}\right] \\ W &= \frac{3}{8}\rho L w g H^2 = \frac{3}{8}\rho L w g H H. \end{aligned}$$

- Similarly if the tank is on its height,

$$W = \frac{3}{8}\rho H w g L^2 = \frac{3}{8}\rho L w g H L.$$

- And, if it is on its width

$$W = \frac{3}{8}\rho L H g w^2 = \frac{3}{8}\rho L w g H w.$$

Since the green part is the same for all the cases, the most work will be done for the largest dimension, in this case L.