Common trigonometric substitutions

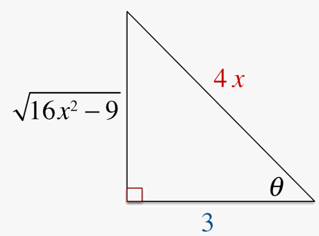
Integral contains:	Substitution	Limit assumptions	Trigonometric identity	Example
$\sqrt{b^2x^2 - a^2}$	$x = \frac{a}{b}\sec\theta$	$0 \le \theta < \frac{\pi}{2},\ \frac{\pi}{2} < \theta \le \pi$	$\sec^2\theta - 1 = \tan^2\theta$	Example 1
$\sqrt{a^2 - b^2x^2}$	$x = \frac{a}{b}\sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$	Example 2
$\sqrt{a^2 + b^2x^2}$	$x = \frac{a}{b}\tan\theta$	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$	$\tan^2\theta + 1 = \sec^2\theta$	Example 3

Example 1

Find the integral $$\int \frac{\sqrt{16x^2 - 9}}{x}\,dx.$$

In this case $a = 3$ and $b = 4$, and our substitution should be

$$x = \frac{3}{4}\sec\theta, \qquad dx = \frac{3}{4}\sec\theta\tan\theta\,d\theta, \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{4x}{3}$$



with this the square root term becomes,

$$\sqrt{16x^2 - 9} = \sqrt{16\left(\frac{9}{16}\sec^2\theta\right) - 9} = \sqrt{9\left(\sec^2\theta - 1\right)} = 3\sqrt{\tan^2\theta} = 3\tan\theta,$$

which gives the integrals

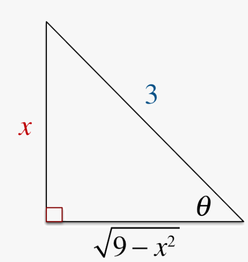
$$\begin{aligned}\int \frac{\sqrt{16x^2 - 9}}{x}\,dx = \int \frac{3\tan\theta}{\frac{3}{4}\sec\theta}\cdot\left(\frac{3}{4}\sec\theta\tan\theta\,d\,\theta\right) &= 3\int \tan^2\theta\,d\theta \\ &= 3\int\left(\sec^2\theta - 1\right)\,d\theta \\ &= 3\left(\tan\theta - \theta\right) + C \\ &= 3\left[\frac{\sqrt{16x^2 - 9}}{3} - \sec^{-1}\left(\frac{4x}{3}\right)\right] + C.\end{aligned}$$

Example 2

Find the integral $$\int \frac{1}{x^2\sqrt{9 - x^2}}\,dx.$$

In this case $a = 3$ and $b = 1$, and our substitution should be

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\,d\theta, \qquad \sin\theta = \frac{x}{3}$$



with this the square root term becomes,

$$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9\left(1 - \sin^2\theta\right)} = 3\sqrt{\cos^2\theta} = 3\cos\theta.$$

Common trigonometric substitutions, cont.

Example 2, cont.

We obtain the integral,

$$\int \frac{1}{x^2\sqrt{9-x^2}}\,dx = \int \frac{1}{(9\sin^2\theta)(3\cos\theta)}(3\cos\theta\,d\theta) = \int \frac{1}{9\sin^2\theta}\,d\theta$$

$$= \frac{1}{9}\int \csc^2\theta\, d\theta = -\frac{1}{9}\cot\theta + C = -\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + C.$$

Example 3

Find the integral $$\int \frac{x}{\sqrt{2x^2-4x+11}}\,dx.$$

Here we first need to complete the square inside the square root and then do the trigonometric substitution:

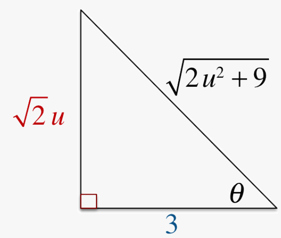
$$2x^2-4x+11 = 2\left(x^2-2x+\frac{11}{2}\right) = 2\left(x^2-2x+1-1+\frac{11}{2}\right) = 2\left((x^2-2x+1)+\frac{9}{2}\right) = 2\left((x-1)^2+\frac{9}{2}\right).$$

For clarity, let's do a u-substitution with $u = x-1$ and $du = dx$,

$$\int \frac{x}{\sqrt{2x^2-4x+11}}\,dx = \int \frac{x}{\sqrt{2(x-1)^2+9}}\,dx = \int \frac{u+1}{\sqrt{2u^2+9}}\,du.$$

Here $a = 3$ and $b = \sqrt{2}$, so that we obtain the trigonometric substitution,

$$u = \frac{3}{\sqrt{2}}\tan\theta, \qquad du = \frac{3}{\sqrt{2}}\sec^2\theta\,d\theta, \qquad \tan\theta = \frac{\sqrt{2}\,u}{3}$$



with this the square root term becomes,

$$\sqrt{2u^2+9} = \sqrt{2\left(\frac{9}{2}\tan^2\theta\right)+9} = \sqrt{9\tan^2\theta+9} = \sqrt{9\left(\tan^2\theta+1\right)} = 3\sqrt{\sec^2\theta} = 3\sec\theta,$$

and the integral becomes

$$\int \frac{x}{\sqrt{2x^2-4x+11}}\,dx = \int \frac{u+1}{\sqrt{2u^2+9}}\,du = \int \frac{\frac{3}{\sqrt{2}}\tan\theta+1}{3\sec\theta}\left(\frac{3}{\sqrt{2}}\sec^2\theta\,d\theta\right)$$

$$= \frac{3}{2}\int \tan\theta\sec\theta\,d\theta + \frac{1}{\sqrt{2}}\int \sec\theta\,d\theta$$

$$= \frac{3}{2}\sec\theta + \frac{1}{\sqrt{2}}\ln|\sec\theta+\tan\theta| + C$$

$$= \frac{3}{2}\frac{\sqrt{2u^2+9}}{3} + \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2u^2+9}}{3}+\frac{\sqrt{2}u}{3}\right| + C$$

$$= \frac{3}{2}\frac{\sqrt{2(x-1)^2+9}}{3} + \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2(x-1)^2+9}}{3}+\frac{\sqrt{2}(x-1)}{3}\right| + C.$$