## Products of powers of sine and cosines

Find the integral

$$\int \sin^m x \cos^n x\, dx.$$

### Using Pythagorean identity $\cos^2 x + \sin^2 x = 1$

#### 1. $m$ is odd

Use the identity $\sin^2 x = 1 - \cos^2 x$

**Example**

Find the integral $\int \sin^5(2x)\cos^3(2x)\, dx$

We first use a $u$-substitution $u = 2x$, $du = 2dx$, to obtain

$$\frac{1}{2}\int \sin^5 u \cos^3 u\, du$$

To solve the integral

$$\begin{aligned}
\frac{1}{2}\int \sin^5 u \cos^3 u\, du &= \frac{1}{2}\int \sin^4 u \cdot \sin u \cos^3 u\, du = \frac{1}{2}\int \left(\sin^2 u\right)^2 \sin u \cos^3 u\, du \\
&= \frac{1}{2}\int \left(1 - \cos^2 u\right)^2 \sin u \cos^3 u\, du \\
&= \frac{1}{2}\int \left(1 - 2\cos^2 u + \cos^4 u\right) \sin u \cos^3 u\, du \\
&= \frac{1}{2}\int \left(\cos^3 u - 2\cos^5 u + \cos^7 u\right) \sin u\, du
\end{aligned}$$

One more $u$-substitution, with $v = \cos u$ and $dv = -\sin u\, du$, gives

$$\begin{aligned}
-\frac{1}{2}\int \left(v^3 - 2v^5 + v^7\right)\, dv &= -\frac{1}{2}\left(\frac{v^4}{4} - 2\frac{v^6}{6} + \frac{v^8}{8}\right) + C = \frac{v^6}{6} - \frac{v^4}{8} - \frac{v^8}{16} + C \\
&= \frac{1}{6}\cos^6 u - \frac{1}{8}\cos^4 u - \frac{1}{16}\cos^8 u + C \\
&= \frac{1}{6}\cos^6(2x) - \frac{1}{8}\cos^4(2x) - \frac{1}{16}\cos^8(2x) + C
\end{aligned}$$

#### 2. $m$ is even and $n$ is odd

Use the identity $\cos^2 x = 1 - \sin^2 x$ and follow the same procedure as the previous case.

### Using double angle identities

When both $m$ and $n$ are even, we use the following identities

$$\sin^2 x = \frac{1 - \cos(2x)}{2}, \qquad \cos^2 x = \frac{1 + \cos(2x)}{2}.$$

## Products of powers of sine and cosines, cont.

### Example

Find the integral $\int 8\sin^4 y\cos^2 y\,dy$

Substitution gives

$$\begin{aligned}
\int 8\left(\sin^2 y\right)^2\cos^2 y\,dy &= \int 8\left(\frac{1-\cos(2y)}{2}\right)^2\left(\frac{1+\cos(2y)}{2}\right)dy\\
&= \int 8\left(\frac{1-2\cos(2y)+\cos^2(2y)}{4}\right)\left(\frac{1+\cos(2y)}{2}\right)dy\\
&= \int \left(1+\cos(2y)-2\cos(2y)-2\cos^2(2y)+\cos^2(2y)+\cos^3(2y)\right)dy\\
&= \int \left(1-\cos(2y)-\cos^2(2y)+\cos^3(2y)\right)dy\\
&= \int dy-\int\cos(2y)\,dy-\int\frac{1+\cos(4y)}{2}dy+\int\left(1-\sin^2(2y)\right)\cos(2y)\,dy\\
&= \int\frac{1}{2}dy-\int\cos(2y)\,dy-\int\frac{1}{2}\cos(4y)\,dy+\int\frac{1}{2}(1-u^2)du\\
&= \frac{1}{2}y-\frac{1}{2}\sin(2y)-\frac{1}{8}\sin(4y)+\frac{1}{2}u-\frac{1}{6}u^3+C\\
&= \frac{1}{2}y-\frac{1}{2}\sin(2y)-\frac{1}{8}\sin(4y)+\frac{1}{2}\sin(2y)-\frac{1}{6}\sin^3(2y)+C
\end{aligned}$$

## Eliminating square roots

We can use the Pythagorean and double angle identities to simplify square roots, as in the two examples below.

### Example 1

Find the integral $\int\theta\sqrt{1-\cos(2\theta)}\,d\theta$

Using the double angle identity we obtain

$$\int\theta\sqrt{1-\cos(2\theta)}\,d\theta = \int\theta\sqrt{2\sin^2\theta}\,d\theta=\sqrt{2}\int\theta\sin\theta d\theta,$$

Integration by parts gives

$$\begin{aligned}
u &= \theta, & du &= d\theta\\
dv &= \sin\theta d\theta, & v &= -\cos\theta
\end{aligned}$$

$$\begin{aligned}
\sqrt{2}\int\theta\sin\theta d\theta &= -\theta\cos\theta+\int\cos\theta d\theta\\
&= -\theta\cos\theta+\sin\theta+C
\end{aligned}$$

## Eliminating square roots, cont.

### Example 2

Find the integral $\int \left(1-\cos^2 t\right)^{3/2} dt$

Using the Pythagorean identity we obtain

$$\begin{aligned}\int \left(1-\cos^2 t\right)^{3/2} dt = \int \left(\sin^2 t\right)^{3/2} dt &= \int \sin^3 t dt \\ &= \int \left(1-\cos^2 t\right) \sin t \, dt \\ &= \int \left(u^2 - 1\right) du \\ &= \frac{1}{3}u^3 - u + C = \frac{1}{3}\cos^3 t - \cos t + C.\end{aligned}$$

## Integrals of sines and cosines

Use integration by parts.

### Example

Find the integral $\int \sin(2x)\cos(3x)\, dx.$

- We use integration by parts with

$$\begin{aligned} u &= \sin(2x), & du &= 2\cos(2x)\, dx \\ dv &= \cos(3x)\, dx, & v &= \frac{1}{3}\sin(3x)\end{aligned}$$

giving

$$\int \sin(2x)\cos(3x)\, dx = \frac{1}{3}\sin(2x)\sin(3x) - \frac{2}{3}\int \cos(2x)\sin(3x)\, dx.$$

- We integrate the last integral by parts with,

$$\begin{aligned} u &= \cos(2x), & du &= -2\sin(2x)\, dx \\ dv &= \sin(3x)\, dx, & v &= -\frac{1}{3}\cos(3x)\end{aligned}$$

to obtain

$$\begin{aligned}\int \sin(2x)\cos(3x)\, dx &= \frac{1}{3}\sin(2x)\sin(3x) - \frac{2}{3}\left[-\frac{1}{3}\cos(2x)\cos(3x) - \frac{2}{3}\int \sin(2x)\cos(3x)\, dx\right] \\ &= \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x) + \frac{4}{9}\int \sin(2x)\cos(3x)\, dx\end{aligned}$$

- We subtract $\frac{4}{9}\int \sin(2x)\cos(3x)\, dx$ on both sides, to obtain

$$\begin{aligned}\frac{5}{9}\int \sin(2x)\cos(3x)\, dx &= \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x) + C \\ \int \sin(2x)\cos(3x)\, dx &= \frac{3}{5}\sin(2x)\sin(3x) + \frac{2}{5}\cos(2x)\cos(3x) + C\end{aligned}$$

## Integrals of powers of $\tan x$ and $\sec x$

We use the identity $1 + \tan^2 x = \sec^2 x$.

### Example

Find the integral $$\int \sec^4 x \tan^2 x \, dx.$$

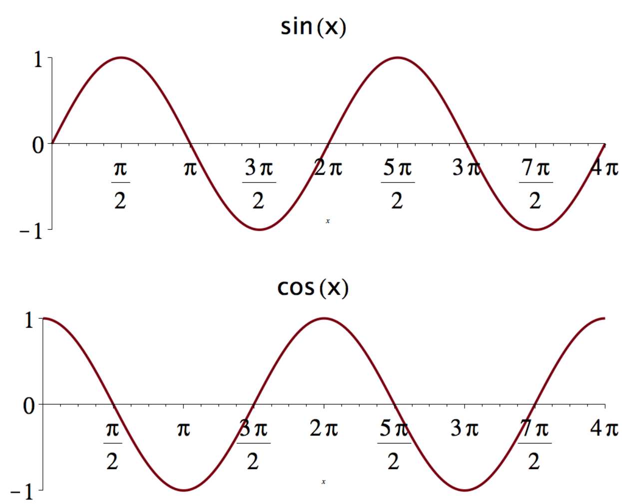
$$\int \sec^4 x \tan^2 x \, dx = \int \sec^2 x \left(1 + \tan^2 x\right) \tan^2 x \, dx = \int \sec^2 x \left(\tan^2 x + \tan^4 x\right) \, dx,$$

using the substitution $u = \tan x$, $du = \sec^2 x \, dx$, we get

$$\begin{aligned} \int \sec^4 x \tan^2 x \, dx &= \int \left(u^2 + u^4\right) \, du \\ &= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C \\ &= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C \end{aligned}$$

## Evaluating definite integrals

To evaluate sines and cosines, it is always useful to remember the plots for sine and cosine,




And the 1-1-2 and 1-2-3 triangles