Taylor Polynomials

If $f(x)$ is a function with all derivatives at $x = a$, then the nth Taylor polynomial of $f(x)$ near $x = a$ is:

$$\begin{aligned} P_n(x) &= f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}\,(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\,(x-a)^k\,. \end{aligned}$$

Taylor Series

The Taylor series of $f(x)$ near $x = a$ is:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}\,(x-a)^n + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\,(x-a)^n\,. \end{aligned}$$

Maclaurin Series

The Maclaurin series of $f(x)$ is the Taylor series generated by $f(x)$ at $x = 0$:

$$\begin{aligned} f(0) &\approx f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n)}(0)}{n!}\,x^n + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\,x^n. \end{aligned}$$

Taylor Formula

If $f(x)$ has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I

$$\begin{aligned} f(x) &= f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}\,(x-a)^n + R_n(x) \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\,(x-a)^k + R_n(x) = P_n(x) + R_n(x), \end{aligned}$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{n+1} \qquad \text{for some } c \text{ between } a \text{ and } x.$$

Taylor Inequality

If $\left|f^{(n+1)}(x)\right| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}\,|x-a|^{n+1} \qquad \text{for } |x-a| \le d$$

Binomial Series

For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \left(\begin{array}{c} m \\ k \end{array} \right) x^k, \quad \text{where} \quad \left(\begin{array}{c} m \\ k \end{array} \right) = \frac{m\,(m-1)\,(m-2)\cdots(m-k+1)}{k!}.$$

Examples of Taylor Series

- $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$ $\qquad -1 < x < 1$
- $\ln|x+1| = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots (-1)^{n+1)} \dfrac{x^n}{n} + \cdots$ $\qquad -1 < x \leq 1$
- $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots + \dfrac{x^n}{n!} + \cdots$ $\qquad -\infty < x \leq \infty$
- $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} - \cdots (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} + \cdots$ $\qquad -\infty < x \leq \infty$
- $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} + \cdots (-1)^n \dfrac{x^{2n}}{(2n)!} + \cdots$ $\qquad -\infty < x \leq \infty$
- $\tan^{-1} x = \arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \cdots (1-)^n \dfrac{x^{2n+1}}{(2n+1)} + \cdots$ $\qquad -1 \leq x \leq 1$

Example

1. What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

when $-0.3 \leq x \leq 0.3$?

In order to compute the error bound, follow these steps.

- Compute the $(n+1)$th derivative of $f(x)$
- Find the upper bound on $f^{(n+1)}(z)$ for $z \in [a, x]$
- Compute $R_n(x)$

Solution:
We know the Taylor (Maclaurin) series for $\sin x$ is: $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

then the approximation can be expressed as $\quad \sin x = P_5(x) + R_5(x)$ i.e., $n = 5$.

- $f^{(6)}(x) = -\sin x$
- $\left| f^{(6)}(x) \right| = |-\sin x| \leq 1$, for all x. This means $M = 1$ in the Taylor Inequality
- $|R_5(x)| \leq M \dfrac{|x|^6}{6!} = \dfrac{|x|^6}{6!} \leq \dfrac{|0.3|^6}{6!}$