Definition

Given a sequence of numbers $a_1,\, a_2, \ldots, a_n, \ldots$ (denoted $\{a_n\}$), we can form sums:

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

obtained by summing the first n numbers in the sequence. We call s_n the nth partial sum of the sequence. Letting $n \to \infty$ result in the *infinite series*

$$\sum_{n=1}^{\infty} a_n.$$

If the sequences of partial sums $\{s_1, s_2, \ldots, s_n, \ldots\}$ converges to a limit L, we say the series **converges** and that its **sum** is L:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots = L.$$

Combining series

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

- $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n = A \pm B$
- $\sum c\, a_n = c \sum a_n = c\, A$ for any constant c

The nth-term test for divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exists or is different from zero.

This test is inconclusive if $\lim_{n\to\infty} a_n = 0$; that is, we cannot say whether the series is convergent or divergent

Geometric series

Geometric series are of the form

$$a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^{n-1} + \cdots = \sum_{n=1}^{\infty} a\,r^{n-1}$$

If $|r| < 1$, the geometric series **converges** to

$$\sum_{n=1}^{\infty} a\,r^{n-1} = \frac{a}{1-r}$$

If $|r| \geq 1$, the series **diverges**.

The integral test

Let $\{a_n\}$ be a sequence of **positive** terms. Supposed that $a_n = f(n)$, where f is a **continuous**, **positive**, **decreasing** function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

Comparison tests

Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with **nonnegative** terms. Suppose that for some integer N

$$d_n \leq a_n \leq c_n, \qquad \text{for all} \quad n > N$$

- If $\sum c_n$ converges, then $\sum a_n$ also converges
- If $\sum d_n$ diverges, then $\sum a_n$ also diverges

Limit comparison tests

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ **and** $\sum b_n$ converges, then $\sum a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ **and** $\sum b_n$ diverges, then $\sum a_n$ diverges.

Absolute convergence

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Note that we cannot say anything about divergence.

The ratio test

Suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

- If $L < 1$ the series $\sum a_n$ converges absolutely
- If $L > 1$ the series $\sum a_n$ diverges
- If $L = 1$ this test gives no information

The root test

Suppose that

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

- If $L < 1$ the series $\sum a_n$ converges absolutely
- If $L > 1$ the series $\sum a_n$ diverges
- If $L = 1$ this test gives no information

Alternating series

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a non-increasing sequence of positive numbers and $\lim_{n\to\infty} a_n = 0$. Then the following alternating series converges,

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$