Geometric series

We can analyze all geometric series simultaneously by allowing r to vary, that is we can consider the following function of x

$$\sum_{n=0}^{\infty} a\, x^n = \frac{a}{1-x}.$$

From previous sections we know these series converges if $|x| < 1$ and diverges for $|x| \geq 1$.

Here we want to allow for more general coefficients, instead of the same coefficient for all the values of n,

$$\sum_{n=0}^{\infty} a_n\, x^n,$$

with the understanding that a_n may depend on n **but not on x.**

Power series

A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1\, x + a_2\, x^2 + \cdots + a_n\, x^n + \cdots$$

A power series about $x = c$ is a series of the form

$$\sum_{n=0}^{\infty} a_n\, (x-c)^n = a_0 + a_1\, (x-c) + a_2\, (x-c)^2 + \cdots + a_n\, (x-c)^n + \cdots$$

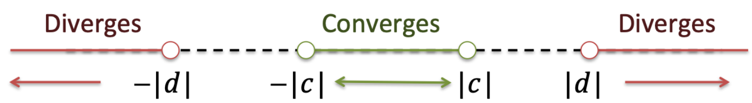
Radius and interval of convergence

A power series $\sum a_n\, x^n$ will converge only for certain values of x. In general, there is always and interval $(-R, R)$ in which a power series converges, and the number R is called the radius of convergence, while the interval itself is called the interval of convergence.

Convergence for power series

A power series always converges absolutely within its radius of convergence.

If the power series $\sum_{n=0}^{\infty} a_n\, x^n$ converges at $x = c \neq 0$, then it converges absolutely for all $-|c| < x < |c|$. If the series diverges at $x = d$, then it diverges for all $x > |d|$ or $x < -|d|$.



How to test a power series for convergence

1. Use the Ratio test (or Root test) to find the interval where the series converges absolutely

$$|x-c| < R \qquad \text{or} \qquad c - R < x < c + R$$

2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. Use Comparison Test, Integral Test, or Alternating Series Test.

3. If the interval of absolute convergence is $|x-c| < R$, the series diverges for $|x-c| > R$.

Calculus with power series

Suppose the power series

$$f(x) = \sum_{n=0}^{\infty} a_n\,(x-c)^n$$

has radius of convergence R. Then

$$f'(x) = \sum_{n=1}^{\infty} n\,a_n\,(x-c)^{n-1}$$

and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}\,(x-c)^{n+1}$$

and these two series have radius of convergence R as well.

Example

Evaluate the convergence of

$$\sum_{n=0}^{\infty} \frac{n\,(x+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} b_n$$

For this series $c=-2$ and $a_n = \dfrac{n}{3^{n+1}}$.

1. Find R using the Ratio Test

$$\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = \lim_{n\to\infty} \frac{\left|\frac{(n+1)\,(x+2)^{n+1}}{3^{n+2}}\right|}{\left|\frac{n\,(x+2)^n}{3^{n+1}}\right|} = \lim_{n\to\infty} \frac{\left|(n+1)\,(x+2)^{n+1}\,3^{n+1}\right|}{\left|n\,(x+2)^n\,3^{n+2}\right|} = \lim_{n\to\infty} \frac{n+1}{n}\left|\frac{x+2}{3}\right| = \left|\frac{x+2}{3}\right|.$$

For convergence we need

$$\left|\frac{x+2}{3}\right| < 1 \qquad \Rightarrow \qquad |x+2| < 3$$

$$\boxed{\text{The radius of convergence is } \mathbf{R=3}.}$$

2. Find the interval of convergence

$$\begin{aligned} & |x+2| & < 3 \\ -3 < \; & x+2 & < 3 \\ -5 < \; & x & < 1. \end{aligned}$$

And we need to evaluate convergence at the end points.

When $x=-5$ the series is $\displaystyle\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\,n$, which diverges since $\lim_{n\to\infty} n = \infty$

When $x=1$ the series is $\displaystyle\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} n$, which diverges since $\lim_{n\to\infty} n = \infty$

$$\boxed{\text{The interval of convergence is } \mathbf{(-5,1)}.}$$

3. For what values of x does this series diverge?

$$\boxed{\text{The series diverges for } \mathbf{(-\infty,5]\cup[1,\infty)}.}$$