## Parametric equations

For certain applications it is not possible to express the relationship between different variables as $y = f(x)$ or $x = g(y)$, the most common example is the equation of a circle centered at $(a, b)$ with radius $r$.

$$(x-a)^2 + (y-b)^2 = r^2$$

To deal with these types of problems we introduce **parametric equations**, for this we define $x$ and $y$ as functions of a third variable, $t$,

$$x = f(t) \qquad y = g(t)$$

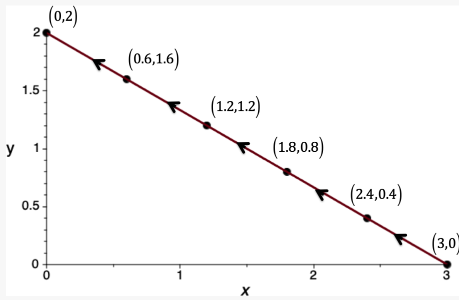
By varying $t$ we can define a set of points $(f(t), g(t))$. The graph of that collection of points is called a **parametric curve**.

### Example

Sketch the parametric curve for the following set of parametric equations

$$x = 3 - 3t, \qquad y = 2t, \qquad 0 \leq t \leq 1$$

| $t$ | $x$ | $y$ |
|---|---|---|
| 0 | 3 | 0 |
| 0.2 | 2.4 | 0.4 |
| 0.4 | 1.8 | 0.8 |
| 0.6 | 1.2 | 1.2 |
| 0.8 | 0.6 | 1.6 |
| 1 | 0 | 2 |




## From parametric to algebraic equations

Sometimes we can eliminate the parameter $t$ and obtain an equation in terms of $y$ and $x$, we call this the algebraic equation to differentiated from the original parametric equation.

### Example

Continuing with our previous example

$$x = 3 - 3t, \qquad y = 2t, \qquad 0 \leq t \leq 1$$

We can solve for $t$ in one equation and substitute in the other:

$$\begin{aligned} t &= 1 - \frac{x}{3} \\ y &= 2t = 2\left(1 - \frac{x}{3}\right) \\ y &= -\frac{2}{3}x + 2. \end{aligned}$$

Which we recognize as the equation of the line with $y$-intersect $(0, 2)$ and slope $-2/3$.

**We have to be careful here, because from the parametric equation this is not defined for all $x$ and $y$. The correct answer should be**

$$y = -\frac{2}{3}x + 2, \qquad 0 \leq x \leq 3, \qquad 0 \leq y \leq 2$$

## Orientation

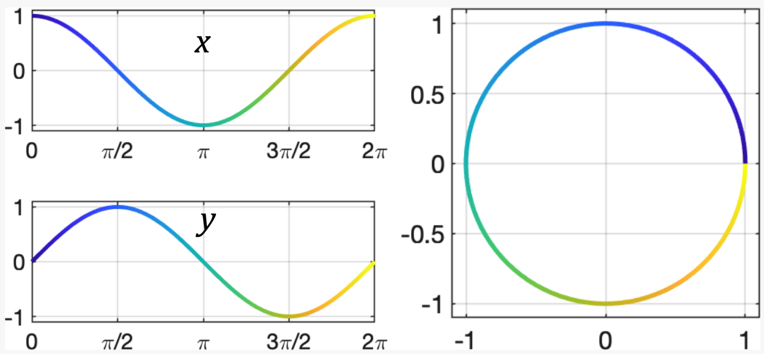
If we consider $t$ to represent time and $x$-$y$ to represent the position of a particle in the plane at different times, then it is easy to see that the parametric curve has a "orientation," roughly this would be the direction in which $t$ is increasing. This means that two parametric equations can look alike, but differ from one another on their orientation.

### Example

Find the parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2$.

*The parametric equations for a circle are*

$$x(t) = a\cos(t), \qquad y(t) = a\sin(t), \qquad 0 \le t \le 2\pi$$

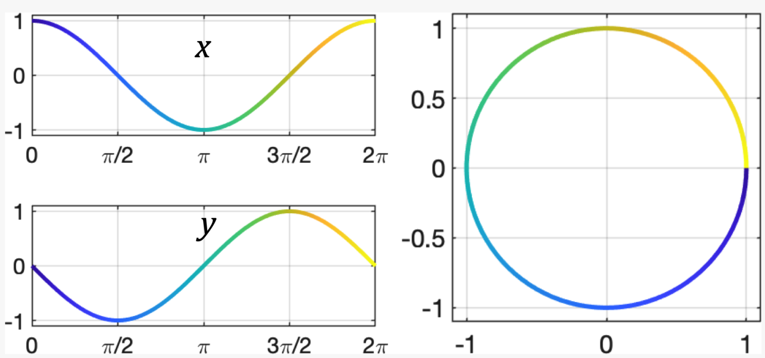



1. **The particle goes around the circle once counterclockwise**

   Here $x$ is going from 1 to 0 to -1 to 0, while $y$ is going 0 to 1 to 0 to -1. The parametric equations are

   $$x(t) = a\cos(t), \qquad y(t) = a\sin(t), \qquad 0 \le t \le 2\pi$$

2. **The particle goes around the circle once clockwise**




   Here $x$ is going from 1 to 0 to -1 to 0, while $y$ is going 0 to -1 to 0 to 1. The parametric equations are

   $$x(t) = a\cos(t), \qquad y(t) = -a\sin(t), \qquad 0 \le t \le 2\pi$$

3. **The particle goes around the circle twice clockwise**

   While $x$ and $y$ still follow the equations from part 2, the range of $t$ changes and the parametric equations are

   $$x(t) = a\cos(t), \qquad y(t) = -a\sin(t), \qquad 0 \le t \le 4\pi$$