Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2}{(x-1)^3},$$

with the following initial condition

$$y(2) \quad = \quad 1.$$

To solve the ODE we need to separate terms with y from terms with x

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2}{(x-1)^3} \\ dy &= \frac{x^2}{(x-1)^3}\,dx \end{aligned}$$

and integrate both sides

$$\int dy \quad = \quad \int \frac{x^2}{(x-1)^3}\,dx$$

Here we focus on the integral on the right hand side. to solve we use partial fractions:

$$\frac{x^2}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

Multiplying both sides by $(x-1)^3$ gives

$$\begin{aligned} x^2 &= A(x-1)^2 + B(x-1) + C \\ &= Ax^2 - 2Ax + A + Bx - B + C. \end{aligned}$$

The resulting system of equations for A, B and C is

$$\begin{aligned} 1 &= A \\ 0 &= -2A + B \\ 0 &= A - B + C \end{aligned}$$

with solution $A = 1$, $B = 2$ and $C = 1$.

Then our integral reduces to

$$\int \frac{x^2}{(x-1)^3}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{2}{(x-1)^2}\,dx + \int \frac{1}{(x-1)^3}\,dx$$

We can solve all three integrals using the substitution $u = x - 1$, $du = dx$

$$\begin{aligned} \int \frac{x^2}{(x-1)^3}\,dx &= \int \frac{1}{x-1}\,dx + \int \frac{2}{(x-1)^2}\,dx + \int \frac{1}{(x-1)^3}\,dx \\ &= \int \frac{du}{u} + \int \frac{2\,du}{u^2} + \int \frac{du}{u^3} \\ &= \ln|u| - 2\frac{1}{u} - \frac{1}{2u^2} + C \\ &= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2(x-1)^2} + C. \end{aligned}$$

Now we can find the solution of our ODE

$$\begin{aligned} \int dy &= \int \frac{x^2}{(x-1)^3}\,dx \\ y &= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2(x-1)^2} + C. \end{aligned}$$

To find the constant we use the initial condition $y(2) = 1$

$$\begin{aligned} y &= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2(x-1)^2} + C \\ y(2) &= \ln|2-1| - \frac{2}{2-1} - \frac{1}{2(2-1)^2} + C \\ 1 &= \ln(1) - \frac{2}{1} - \frac{1}{2\cdot 1} + C \\ 1 &= 0 - 2 - \frac{1}{2} + C \end{aligned}$$

solving for C gives $C = \frac{7}{2}$, and the solution of our ODE is

$$\boxed{y = \ln|x-1| - \frac{2}{x-1} - \frac{1}{2(x-1)^2} + \frac{7}{2}.}$$

Exercise

1. What will be the solution if the initial condition is $y(2) = -1$?

   In this case

$$\begin{aligned} y(2) &= \ln|2-1| - \frac{2}{2-1} - \frac{1}{2(2-1)^2} + C \\ -1 &= \ln(1) - \frac{2}{1} - \frac{1}{2\cdot 1} + C \\ -1 &= 0 - 2 - \frac{1}{2} + C \Rightarrow C = \frac{3}{2} \end{aligned}$$

$$\boxed{y = \ln|x-1| - \frac{2}{x-1} - \frac{1}{2(x-1)^2} + \frac{3}{2}.}$$

2. Without solving, what would be the procedure if the numerator of the ODE is x^4, instead of x^2?

   In this case we have an improper fraction and we use long division

$$\begin{array}{r|l} & x+3 \\ \hline x^3-3x^2+3x-1 & x^4 \\ & -x^4+3x^3-3x^2+x \\ \hline & 3x^3-3x^2+x \\ & -3x^3+9x^2-9x+3 \\ \hline & 6x^2-8x+3 \end{array}$$

$$\frac{x^4}{(x-1)^3} = x + 3 + \frac{6x^2-8x+3}{(x-1)^3}$$