Integration by parts formula

$$\int u\,dv = u\,v - \int v\,du$$
$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

This technique is most useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. As a rule of thumb, you must choose $f(x)$ such that its derivative is simpler than $f(x)$.

Some things to considering when integrating by parts

1. LIATE

In *most* cases it is convenient to set $f(x)$ (or u) to be a function on this list

- **L**ogarithm
- **I**nverse trigonometric function
- **A**lgebraic function
- **T**rigonometric function
- **E**xponential

2. Integrate more than once

Some times you will need to keep integrating by parts until the resulting integral is simple enough.

Example

Find the integral $\int x^2 e^x dx,$

To apply LIATE note that we choose u to be the algebraic function, since **E**xponential functions come after **A**lgebraic function in the LIATE scheme,

$$u = x^2, \qquad du = 2x dx$$
$$dv = e^x dx \qquad v = e^x,$$

so that

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx,$$

here we will need to apply integration by parts to the last integral one more time,

$$\int x^2 e^x dx = x^2 e^x - \int 2x\, e^x dx,$$

$$\int x^2 e^x dx = x^2 e^x - \left[2x\, e^x - \int 2e^x dx\right] = x^2 e^x - 2xe^x + \int 2e^x dx,$$

we can solve the last integral easily, giving us the solution

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C.$$

3. Tabular Method

In the last example we had to apply the Integration by Parts Formula multiple times. There is a convenient way to simplify our work. This is done by creating a table as illustrated in the following example.

Example

Find the integral

$$\int x^3 \sin(3x)dx,$$

following LIATE, $u = x^3$ and $v = \sin(3x)$, so that we need to *differentiate* u and *integrate* v, we perform these operations in a table .

Differentiate u		Integrate v
x^3		$\sin(3x)$
$3x^2$	+	$-\frac{1}{3}\cos(3x)$
$6x$	-	$-\frac{1}{9}\sin(3x)$
6	+	$\frac{1}{27}\cos(3x)$
0	-	$\frac{1}{81}\sin(3x)$

Then we "follow" the arrows to obtain our integral:

$$\begin{aligned}\int x^3 \sin(3x)dx &= +(x^3)\left(-\frac{1}{3}\cos(3x)\right) - (3x^2)\left(-\frac{1}{9}\sin(3x)\right) + (6x)\left(\frac{1}{27}\cos(3x)\right) - (6)\left(\frac{1}{81}\sin(3x)\right)\\ &= -\frac{x^3}{3}\cos(3x) + \frac{x^2}{3}\sin(3x) + \frac{2x}{9}\cos(3x) - \frac{2}{27}\sin(3x).\end{aligned}$$

4. $dv = dx$

Sometimes is useful to consider $dv = dx$.

Example

Find the integral

$$\int \ln x dx.$$

Taking

$$u = \ln x, \qquad du = \frac{1}{x}dx$$
$$dv = 1dx, \qquad v = x,$$

gives

$$\int \ln x\, dx = \ln x \cdot x - \int \left[\frac{1}{x}\cdot x\right] dx = x\ln x - \int dx = x\ln x - x + C.$$

5. Recurring integrals

Sometimes you need to reorganize terms by looking at terms that are repeated through the integration process.

Example

Find the integral

$$\int e^x \sin x dx,$$

if we apply LIATE, we get

$$\begin{aligned} u &= \sin x, & du &= \cos x dx \\ dv &= e^x dx & v &= e^x, \end{aligned}$$

so that

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx,$$

we apply LIATE again to the last part

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx,$$

obtaining

$$\begin{aligned} \int e^x \sin x dx &= e^x \sin x - \left[e^x \cos x + \int e^x \sin x dx \right] \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x dx \\ \text{(add the last integral to both sides)} &\Rightarrow \\ 2\int e^x \sin x dx &= e^x \sin x - e^x \cos x \\ \text{(dividing by 2 on both sides)} &\Rightarrow \\ \int e^x \sin x dx &= \frac{e^x \sin x - e^x \cos x}{2}, \end{aligned}$$

since this is an indefinite integral DO NOT FORGET the constant of integration!!

$$\int e^x \sin x dx = \frac{e^x \sin x - e^x \cos x}{2} + C.$$

6. General remarks

- To choose between u and dv, if you only know how to integrate only one of the two, that's the one you integrate!
- When in doubt integrate by parts.
- Recurring integrals comes up often when we are dealing with the product of two functions with "non-terminating" derivatives. By this we mean that you can keep differentiating functions like e^x and trigonometric functions indefinitely and never reach zero. Polynomials on the other hand will eventually "terminate" and their nth derivative (where n is the degree of the polynomial) is identically zero.