If the limit exists, the integral is said to **converge**, otherwise the integral **diverges**.

## Infinite limits of integration

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \underbrace{\int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx}_{\text{solve each of these as described above}}$$

## Discontinuities in $[a, b]$

- Discontinuity at $x = a$

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

- Discontinuity at $x = b$

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

- Discontinuity at $x = c, \quad a \le c \le b$

$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx + \int_c^b f(x)\,dx}_{\text{solve each of these as described above}}$$

## Examples

### Example 1

$$\int_1^\infty \frac{1}{x^{3/2}}\,dx = \lim_{t\to\infty}\left[\int_1^t x^{-3/2}\,dx\right] = \lim_{t\to\infty}\left[-2x^{-1/2}\right]_1^t = \lim_{t\to\infty}\left[-2\frac{1}{\sqrt{t}}\right] - \left(-\frac{2}{\sqrt{1}}\right) = 0 + 2 = 2.$$

**Convergent**

### Example 2

$$\int_0^1 \frac{1}{x^{3/2}}\,dx = \lim_{t\to 0}\left[\int_t^1 x^{-3/2}\,dx\right] = \lim_{t\to 0}\left[-2x^{-1/2}\right]_t^1 = -\frac{2}{\sqrt{1}} - \lim_{t\to 0}\left[-2\frac{1}{\sqrt{t}}\right] = -2 + \infty = \infty.$$

**Divergent**

## Convergence tests: $p-$test

1. For $0 < a < \infty$

$$\int_a^{\infty} \frac{1}{x^p}\,dx \quad \begin{cases} \text{converges} & if \quad p > 1 \\ \text{diverges} & if \quad p \leq 1 \end{cases}$$

2. For $c \leq 0 < b < \infty$

$$\int_c^{b} \frac{1}{x^p}\,dx \quad \begin{cases} \text{converges} & if \quad p < 1 \\ \text{diverges} & if \quad p \geq 1 \end{cases}$$

3. For $-\infty < a < b < \infty$

$$\left.\begin{array}{l} \displaystyle\int_a^b \frac{1}{(x-a)^p}\,dx \\ \displaystyle\int_a^b \frac{1}{(b-x)^p}\,dx \end{array}\right\} \quad \begin{cases} \text{converges} & if \quad p < 1 \\ \text{diverges} & if \quad p \geq 1 \end{cases}$$

4. Regardless of the the value of $p$ the following integral **always diverges**

$$\int_0^{\infty} \frac{1}{x^p}\,dx$$

### Examples

- $$\int_{-1}^{1} \frac{dx}{x^{2/3}} = \underbrace{\int_{-1}^{1} \frac{dx}{x^{2/3}}}_{\text{Case } \mathbf{2} \text{ with } c=-1,\ b=1 \text{ and } p=2/3<1 \quad \Rightarrow \quad \text{convergent}}$$

  We can also check converge by evaluating the integral:

$$\begin{aligned} \int_{-1}^{1} \frac{dx}{x^{2/3}} &= \int_{-1}^{0} \frac{dx}{x^{2/3}} + \int_{0}^{1} \frac{dx}{x^{2/3}} = \lim_{t\to 0}\left[\int_{-1}^{t} \frac{dx}{x^{2/3}}\right] + \lim_{t\to 0}\left[\int_{t}^{1} \frac{dx}{x^{2/3}}\right] \\ &= \lim_{t\to 0}\left[3\,x^{1/3}\right]_{-1}^{t} + \lim_{t\to 0}\left[3\,x^{1/3}\right]_{t}^{1} = \lim_{t\to 0} 3t^{1/3} - \left(3(-1)^{1/3}\right) + 3(1)^{1/3} - \lim_{t\to 0} 3t^{1/3} \\ &= 0+3+3+0 = 6. \end{aligned}$$

- $$\int_{-\infty}^{\infty} \frac{dx}{x^{2/3}}\,dx = \underbrace{\int_{-\infty}^{0} \frac{dx}{x^{2/3}}\,dx}_{\text{Case } \mathbf{4} \ \Rightarrow \ \text{divergent}} + \underbrace{\int_{0}^{\infty} \frac{dx}{x^{2/3}}\,dx}_{\text{Case } \mathbf{4} \ \Rightarrow \ \text{divergent}}$$

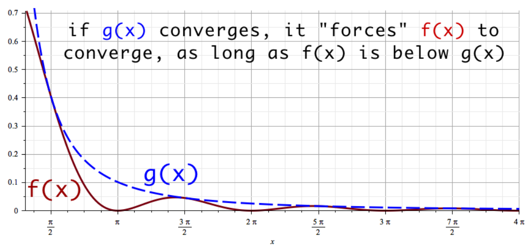
  We can also check converge by evaluating the integral:

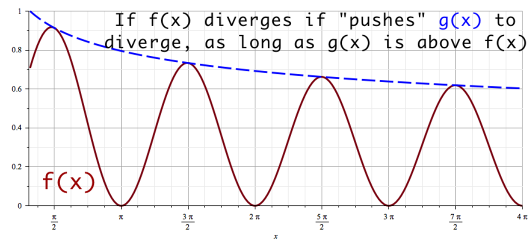
$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{x^{2/3}} &= \lim_{t\to -\infty} \int_{t}^{-1} \frac{dx}{x^{2/3}} + \int_{-1}^{1} \frac{dx}{x^{2/3}} + \lim_{t\to\infty} \int_{1}^{t} \frac{dx}{x^{2/3}} \\ &= 3(-1)^{1/3} - \lim_{t\to -\infty} 3t^{1/3} + 6 + \lim_{t\to\infty} 3t^{1/3} - 3(1)^{1/3} \\ &= \infty. \end{aligned}$$

## Convergence tests: comparison test

Let $f(x)$ and $g(x)$ be two functions defined on $[a,b]$ such that $\quad 0 \leq f(x) \leq g(x) \quad$ for any $x \in [a,b]$, then

- if $\displaystyle\int_a^b g(x)\,dx$ is convergent, then if $\displaystyle\int_a^b f(x)\,dx$ is convergent.
- if $\displaystyle\int_a^b f(x)\,dx$ is divergent, then if $\displaystyle\int_a^b g(x)\,dx$ is divergent.




### Example

Determine of the integral converges or diverges $\displaystyle\int_1^\infty \frac{1}{x^2+5x+6}\,dx$

The integral can be simplified as

$$\int_1^\infty \frac{1}{x^2+5x+6}\,dx = \int_1^\infty \frac{1}{(x+3)(x+2)}\,dx$$

then in the interval $[1,\infty)$

$$\frac{1}{(x+3)(x+2)} < \frac{1}{x+3}$$

And the integral

$$\int_1^\infty \frac{1}{x+3}\,dx$$

is convergent according to the $p-$test. Then using the comparison test we conclude that the integral

$$\int_1^\infty \frac{1}{x^2+5x+6}\,dx$$

is convergent.
Equivalently we can evaluate the integral (using partial fractions)

$$\begin{aligned}
\int_1^\infty \frac{1}{x^2+5x+6}\,dx = \lim_{t\to\infty}\left[\int_1^t \frac{1}{x^2+5x+6}\,dx\right] &= \lim_{t\to\infty}\left[\int_1^t \frac{1}{(x+3)(x+2)}\,dx\right] \\
&= \lim_{t\to\infty}\left[\int_1^t \frac{1}{x+2}\,dx - \int_1^t \frac{1}{x+3}\,dx\right] \\
&= \lim_{t\to\infty}\Big[\ln|x+2| - \ln|x+3|\Big]_1^t = \lim_{t\to\infty}\left[\ln\left|\frac{x+2}{x+3}\right|\right]_1^t \\
&= \lim_{t\to\infty}\left[\ln\left|\frac{t+2}{t+3}\right|\right] - \left[\ln\left|\frac{1+2}{1+3}\right|\right] \\
&= \ln(1) - \ln\left|\frac{3}{4}\right| = -\ln\left|\frac{3}{4}\right|
\end{aligned}$$

## Convergence tests: limit test

Let $f(x)$ and $g(x)$ be two positive functions defined on $[a, b]$. Assume that both functions exhibit an improper behavior at $x = a$ and $f(x) \sim g(x)$ when $x \approx a$, then

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(x)\,dx} \textbf{ is convergent if and only if } \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{g(x)\,dx} \textbf{ is convergent.}$$

### Example

Determine of the integral is convergent or divergent $\displaystyle\int_0^1 \frac{1}{\sqrt{x}+x^4}\,dx$

When $x \approx 0$

$$\frac{1}{\sqrt{x}+x^4} \sim \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}}.$$

The integral $\displaystyle\int_0^1 \frac{1}{x^{1/2}}\,dx$ is convergent according to the $p-$test, so that by the limit test our integral is convergent.

## Convergence tests: limit comparison test

Suppose that:

1. Both $f(x)$ and $g(x)$ are continuos on $[a, \infty)$ and $f(x) > 0$ and $g(x) > 0$ on $[a, \infty)$
2. For some finite positive number $L$: $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$

Then

$$\int_a^\infty f(x)\,dx \qquad \text{and} \qquad \int_a^\infty g(x)\,dx$$

behave the same way, that is either both converge or both diverge.

### Example

Determine if the integral converges or diverges $\displaystyle\int_{2.5}^\infty \frac{2x^3-5x^2-4x+3}{3x^5+2x^4+x+1}\,dx$

For convenience, we wish to compare this to some $p-$integral, so that we look for $g(x) = \dfrac{1}{x^p}$

- If we use $p = 5$, the degree of the denominator we obtain

$$\lim_{x\to\infty} \frac{\frac{2x^3-5x^2-4x+3}{3x^5+2x^4+x+1}}{\frac{1}{x^5}} = \lim_{x\to\infty} \frac{2x^3-5x^2-4x+3}{\frac{1}{x^5}\left(3x^5+2x^4+x+1\right)} = \lim_{x\to\infty} \frac{2x^3-5x^2-4x+3}{3+\frac{2}{x}+\frac{1}{x^4}+\frac{1}{x^5}} = \infty$$

- If we use $p = 2$, the degree of the numerator we obtain

$$\begin{aligned}\lim_{x\to\infty} \frac{\frac{2x^3-5x^2-4x+3}{3x^5+2x^4+x+1}}{\frac{1}{x^2}} &= \lim_{x\to\infty} \frac{2x^3-5x^2-4x+3}{\frac{1}{x^2}\left(3x^5+2x^4+x+1\right)} = \lim_{x\to\infty} \frac{2x^3-5x^2-4x+3}{3x^3+2x^2+\frac{1}{x}+\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{\frac{2x^3}{x^3}-\frac{5x^2}{x^3}-\frac{4x}{x^3}+\frac{3}{x^3}}{\frac{3x^3}{x^3}+\frac{2x^2}{x^3}+\frac{1}{x^3}\cdot\frac{1}{x}+\frac{1}{x^3}\cdot\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2-\frac{5}{x}-\frac{4}{x^2}+\frac{3}{x^3}}{3+\frac{2}{x}+\frac{1}{x^4}+\frac{1}{x^5}} = \frac{2}{3}\end{aligned}$$

Since the integral $\displaystyle\int_{2.5}^\infty \frac{1}{x^2}\,dx$ is convergent according to the $p-$test, then our original integral is convergent too.