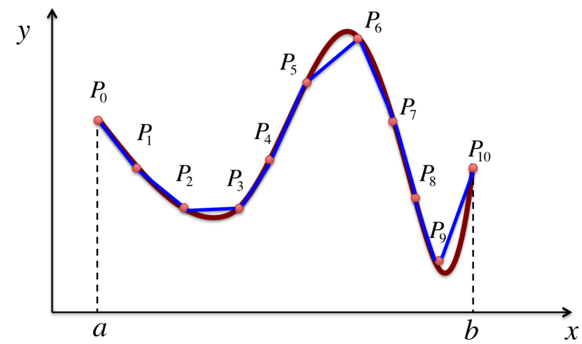
We wish to determine the length of a continuous function $f(x)$ over some interval $[a, b]$. We assume that the derivative of the function is also continuous in this interval.



To estimate the length, we first divide the interval into smaller intervals and approximate the length of the curve as the sum of the length of the lines connecting the subintervals, as shown in the figure.

If we denote the length of the ith line segment as $|P_{i-1} - P_i|$, the toal length of the curve can be approximated by

$$L \approx \sum_{i=1}^{n} |P_{i-1} - P_i| \, .$$

We will get the exact arc length if we use infinitely many subintervals, so that

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1} - P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}.$$

From Calculus I we know that, this is equivalent to $\quad L = \int_a^b \sqrt{dx^2 + dy^2}.$

To find dy, we note that

$$y = f(x) \qquad \Rightarrow \qquad dy = f'(x)dx,$$

where we had made used of the chain rule. The equation for the length now reduces to

$$\begin{aligned} L = \int_a^b \sqrt{dx^2 + dy^2} &= \int_a^b \sqrt{dx^2 + (f'(x))^2 dx^2} \\ &= \int_a^b \sqrt{1 + (f'(x))^2}\, dx. \end{aligned}$$

Sometimes is more convenient to look at this equations in terms of x and y, so that

$$L = \int ds, \qquad \begin{cases} ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} dx & \text{if } y = f(x), \quad a \le x \le b \qquad (1) \\ ds = \sqrt{1 + \left(\dfrac{dx}{dy}\right)^2} dy & \text{if } x = g(y), \quad c \le y \le d \qquad (2) \end{cases}$$

Example: Determine the arc length of the curve $x = \frac{1}{2}y^2$ for $0 \le x \le \frac{1}{2}$.
In this case, differentiating with respect to x gives

$$x = \frac{1}{2}y^2 \qquad \Rightarrow \qquad 1 = \frac{1}{2}2y\frac{dy}{dx} = y\frac{dy}{dx},$$

and our equation for the length becomes

Using Eqn. (1)

$$\begin{aligned} L = \int_0^{\frac{1}{2}} \sqrt{1 + \left(\frac{1}{y}\right)^2} dx &= \int_0^{\frac{1}{2}} \sqrt{1 + \left(\frac{1}{\sqrt{2x}}\right)^2} dx \\ &= \int_0^{\frac{1}{2}} \sqrt{1 + \frac{1}{2x^2}}\, dx. \end{aligned}$$

Using Eqn. (2)

When $x = 0$, $y = 0$ and when $x = \frac{1}{2}$, $y = 1$, so that

$$L = \int_0^1 \sqrt{1 + y^2}\, dy$$

- Solve second equation

$$\int_0^1 \sqrt{1+y^2}\, dy$$

Using the substitution

$$\begin{aligned} y &= \tan\theta, \\ dy &= \sec^2\theta\, d\theta \end{aligned}$$

we get

$$\begin{aligned} \int_0^1 \sqrt{1+y^2}\, dy &= \int \sqrt{1+\tan^2\theta}\, \sec^2\theta\, d\theta = \int \sqrt{\sec^2\theta}\, \sec^2\theta\, d\theta \\ &= \int \sec^3\theta\, d\theta \end{aligned}$$

To solve this lat integral we use integration by parts

$$\begin{aligned} u &= \sec\theta, & du = \tan\theta\sec\theta\, d\theta \\ dv &= \sec^2\theta\, d\theta, & v = \tan\theta \end{aligned}$$

so that

$$\begin{aligned} \int \sec^3\theta\, d\theta = \int u\, dv &= u\, v - \int v\, du \\ &= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta\left(\sec^{\theta} - 1\right)\, d\theta \\ &= \sec\theta\tan\theta - \left[\int \sec^3\theta d\theta - \int \sec\theta d\theta\right] \\ &= \sec\theta\tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta \\ \int \sec^3\theta\, d\theta + \int \sec^3\theta\, d\theta &= \sec\theta\tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta + \int \sec^3\theta\, d\theta \\ 2\int \sec^3\theta\, d\theta &= \sec\theta\tan\theta + \int \sec\theta d\theta \\ \int \sec^3\theta\, d\theta &= \frac{1}{2}\left(\sec\theta\tan\theta + \int \sec\theta d\theta\right) \\ &= \frac{1}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right). \end{aligned}$$

Returning to y

$$\begin{aligned} \int_0^1 \sqrt{1+y^2}\, dy &= \frac{1}{2}\left[\left(\sqrt{1+y^2}\right)\cdot(y) + \ln\left|\left(\sqrt{1+y^2}\right) + (y)\right|\right]\Bigg|_0^1 \\ &= \frac{1}{2}\left[y\sqrt{1+y^2}\Big|_0^1 + \frac{1}{2}\ln\left|y + \sqrt{1+y^2}\right|\Bigg|_0^1\right. \\ &= \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2}). \end{aligned}$$