1. **nth-term test for divergence**

For a series of the form

$$\sum_{n=1}^{\infty} a_n$$

$\lim_{n\to\infty} a_n = 0$		The test is inconclusive
$\lim_{n\to\infty} a_n = \infty$ (or DNE)		The series diverges
$\lim_{n\to\infty} a_n \neq 0$		The series diverges

This test can only check for **divergence**

Examples: Determine whether the following series converge or diverge

- $\sum_{n=1}^{\infty} \frac{2n^2+1}{n^2-2}$

- $\sum_{n=1}^{\infty} \frac{1}{n}$

- $\sum_{n=2}^{\infty} \ln(n)$

- $\sum_{n=1}^{\infty} ne^{-n}$

2. **Geometric Series**

$$\sum_{n=1}^{\infty} a\, r^{n-1}$$

If $|r| < 1$ the series converges to $\dfrac{a}{1-r}$.

if $|r| \geq 1$ the series diverges.

When a series has n only as an exponent, it can be modified to be a geometric series.

Examples: Determine whether the following series converge or diverge and if convergent find its value

- $\displaystyle\sum_{n=0}^{\infty} 9^{-n+2}\, 4^{n+1}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

- $\displaystyle\sum_{n=1}^{\infty} 4^{2n} 3^{-3n}$

3. **Integral Test**

Assume that for a given series we can find a function such that $a_n = f(n)$, then the series $\sum_{n=k}^{\infty} a_n$ and the integral $\int_k^{\infty} f(x)\,dx$ both converge or both diverge.

> Before we use the integral test we need to check whether $f(x)$ is continuous, positive and decreasing. If $f(x)$ doesn't satisfy *every condition* then the integral test cannot be used.

Example:

Determine whether the following series converge or diverge $\sum_{n=3}^{\infty} \frac{3}{n^2 - 3n + 2}$

The corresponding function is $f(x) = \frac{3}{x^2 - 3x + 2} = \frac{3}{\left(\boxed{}\right)\left(\boxed{}\right)}$

(a) The function is discontinuous at $x =$ ☐ and $x =$ ☐,

(b) The function will be negative in the interval ☐,

(c) To determine if the function is decreasing we look for intervals where the first derivative is negative:

$$f'(x) = \boxed{}$$

For the values of x that correspond to the summation, $x \geq 3$, the first derivative is ☐.

Are the three conditions satisfied?

4. p**-Series**

A p-series is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. To determine when this series converges, let's use the integral test.

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^b = \lim_{b\to\infty} \frac{b^{1-p}}{1-p} - \frac{1}{1-p}$$

From this, it is clear that when $p = 1$ the integrals (and the series) will diverge. To understand the other cases ($p < 1$ and $p > 1$) recall that

$$\lim_{t\to\infty} \frac{1}{t^c} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} t^c = \infty,$$

for any **positive** constant c. Since the limit from the integral can be written as

$$\lim_{b\to\infty} b^{1-p},$$

we can conclude that when $1 - p > 0$, the integral diverges and when $1 - p < 0$ the integral converges. This give us the general rules for the convergence of the p-series as summarized below.

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

converges for $p > 1$
diverges for $p \leq 1$

This series are not included in the exam cheat sheet

Examples: Determine whether the following series converge or diverge

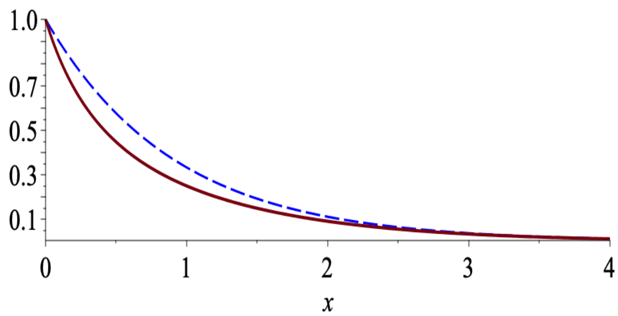
- $\sum_{n=1}^{\infty} \frac{2n+1}{n^3}$

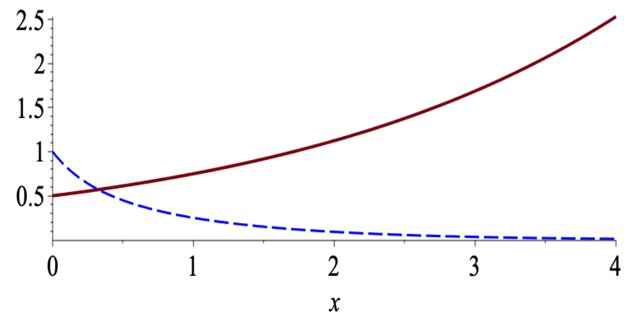
- $\sum_{n=1}^{\infty} n^3 + \frac{1}{n^2}$

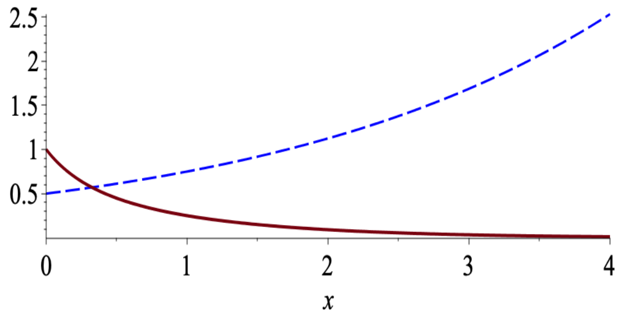
5. **Comparison Test** Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with **nonnegative** terms. Suppose that for some integer N

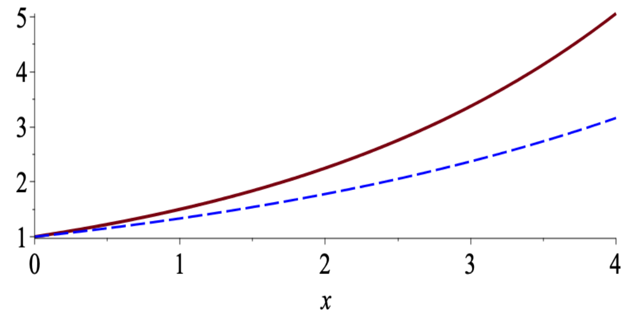
$$d_n \leq a_n \leq c_n, \qquad \text{for all} \quad n > N$$

- If $\sum c_n$ converges, then $\sum a_n$ also converges
- If $\sum d_n$ diverges, then $\sum a_n$ also diverges









The "known" series are usually geometric or p-series

Examples: Determine whether the following series converge or diverge

- $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} + 1\right)^2$

- $\displaystyle\sum_{n=2}^{\infty} \frac{n-1}{\sqrt{n^6+1}}$

6. **Limit comparison tests** Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ **<u>and</u>** $\sum b_n$ converges, then $\sum a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ **<u>and</u>** $\sum b_n$ diverges, then $\sum a_n$ diverges.

> Put the "known" series in the denominator
> If the comparison test is inconclusive, use this test next using the same comparison series

Examples: Determine whether the following series converge or diverge

- $\displaystyle\sum_{n=1}^{\infty} \frac{4n^2 - n}{n^3 + 9}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{2n^2 + 4n + 1}}{n^3 + 9}$

7. **Absolute convergence**

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Note that we cannot say anything about divergence.

Examples: Determine whether the following series converge or diverge

- $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^3+1}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-3}}{\sqrt{n}}$

- $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^{n+1}(n+1)}{n^3+1}$

8. **The ratio test**

Suppose that
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

- If $L < 1$ the series $\sum a_n$ converges absolutely
- If $L > 1$ the series $\sum a_n$ diverges
- If $L = 1$ this test gives no information

Useful when the series involves factorials

Examples: Determine whether the following series converge or diverge

- $\displaystyle\sum_{n=1}^{\infty}\frac{3^{1-2n}}{n^2+1}$

- $\displaystyle\sum_{n=0}^{\infty}\frac{(2n)!}{5n+1}$

- $\displaystyle\sum_{n=3}^{\infty}\frac{e^{4n}}{(n-2)!}$

9. **The root test**

Suppose that

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

- If $L < 1$ the series $\sum a_n$ converges absolutely
- If $L > 1$ the series $\sum a_n$ diverges
- If $L = 1$ this test gives no information

Examples: Determine whether the following series converge or diverge

- $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n+1}{4-2n}\right)^{2n}$

- $\displaystyle\sum_{n=0}^{\infty} \frac{n^{1-3n}}{4^{2n}}$

- $\displaystyle\sum_{n=4}^{\infty} \frac{(-5)^{1+2n}}{2^{5n-3}}$

10. **Alternating series**

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a non-increasing sequence of positive numbers and $\lim_{n\to\infty} a_n = 0$. Then the following alternating series converges,

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

We need to check three things:

(a) The series is alternating, that is we have a series $\sum a_n$ and either

$$a_n = (-1)^n\, b_n \qquad \text{or .} \qquad a_n = (-1)^{n+1}\, b_n$$

where $b_n \geq 0$ for all n.

(b) $\lim_{n\to\infty} b_n = 0$, and

(c) $\{b_n\}$ is a decreasing sequence

If all three conditions are satisfied, the series $\sum a_n$ *converges*

Examples: Determine whether the following series converge or diverge

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7+2n}$

- $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+3}}{n^3+4n+1}$