Find a power series representation of a function

To solve these problems we use the formula for the geometric series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

The strategy is to rearrange the function so that it looks like the right-hand-side of this equation and determine what is the equivalent to r.

Examples: Find the power series representation of the given function

- $f(x) = \dfrac{2}{1-x}$ rearranging gives $f(x) = 2\left[\boxed{}\right]$, in this case $r = \boxed{}$.

  The power series representation is then $f(x) = 2\sum_{n=0}^{\infty} x^n$.

- $f(x) = \dfrac{1}{1-x^2}$, we compare with the formula for geometric series

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$$

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} \boxed{}$$

- $f(x) = \dfrac{1}{4-x}$, before we compare with the formula for geometric series we need to rearrange the series

$$\frac{1}{4-x} = \frac{1}{4}\left[\boxed{}\right]$$

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$$

$$\frac{1}{1-\frac{x}{4}} = \sum_{n=0}^{\infty} \boxed{}$$

  The power series representation is then $f(x) = \boxed{}$.

- $f(x) = \dfrac{1}{1+x}$, before we compare with the formula for geometric series we need to rearrange the series

$$\frac{1}{1+x} = \frac{1}{1-\boxed{}}$$

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$$

$$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty} \boxed{}$$

  The power series representation is then $f(x) = \boxed{}$.

Determine the radius and interval of convergence of a power series

We use the ratio test condition: $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1.$$

Most of the time, the **ratio test** works. *Do not forget the absolute values!*
To find the interval of convergence, *do not forget to check the end points.*

Examples: Determine the radius and interval of convergence of the given power series

- $\sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}} \Rightarrow \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{5^{n+1}}{5^{n+2}}\right| = \lim_{n\to\infty}\left|\boxed{}\right|,$

  taking out all the terms that do not have a n gives: $\left|\boxed{}\right| \lim_{n\to\infty}\boxed{} =$

  We end up with the inequality $\left|\boxed{}\right| < 1$ which can be expanded as $\boxed{} < x < \boxed{}$

  To determine the interval of convergence we need to evaluate convergence at the end points:

  - When $x = -5$ the series becomes $\sum_{n=0}^{\infty}\frac{(-5)^n}{5^{n+1}} = \boxed{}$

  - When $x = 5$ the series becomes $\sum_{n=0}^{\infty}\frac{(5)^n}{5^{n+1}} = \boxed{}$

  Putting everything together gives the interval of convergence $\boxed{}$

  and we can find the radius of convergence as **half** the length of the interval, so that $R = \boxed{}$.

- $\sum_{n=0}^{\infty} n!\, x^n$

- $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)\,3^n}$

Derivatives and Integrals of power series

Since derivatives and integrals are linear operators we can differentiate/integrate term by term

$$\sum_{n=0}^{\infty} 2x^{n+1} = 2x + 2x^2 + 2x^3 + 2x^4 + \cdots$$

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} 2x^{n+1}\right) = \frac{d}{dx}(2x) + \frac{d}{dx}\left(2x^2\right) + \frac{d}{dx}\left(2x^3\right) + \frac{d}{dx}\left(2x^4\right) + \cdots$$

$$= 2 + 4x + 6x^2 + 8x^3 + \cdots = \sum_{n=0}^{\infty} 2(n+1)x^n$$

or equivalently treat the power series as a power rule:

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} 2x^{n+1}\right) = \sum_{n=0}^{\infty} 2\frac{d}{dx}\left(x^{n+1}\right) = \sum_{n=0}^{\infty} 2\left((n+1)\,x^n\right)$$

Same for integration:

$$\int \sum_{n=0}^{\infty} 2x^{n+1}\,dx = \sum_{n=0}^{\infty} 2\int x^{n+1}\,dx = \sum_{n=0}^{\infty} 2\left(\frac{x^{n+2}}{n+2}\right)$$

Example: Find the power series of the given function using derivatives or integrals of known power series

- $f(x) = \ln(1+x)$

  Finding the derivative of $f(x)$ gives:

  $f'(x) =$ [ ] $=$ [ ]

  since we can find the power series representation of $f'(x)$ we have

  $f(x) = \int f'(x)\,dx =$ [ ] $=$ [ ] $+C$

  Finally, we find the value of C by choosing a value of x, for example $x = 0$:

- $\displaystyle\sum_{n=0}^{\infty} \frac{2x}{(1+x)^2}$ Note that : $\displaystyle\frac{d}{dx}\left(\frac{1}{1+x}\right) =$ [ ]