Midterm Exam (with Solutions), SCHC 212, Frank Thorne (thornef@mailbox.sc.edu)

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Please work without books, notes, calculators, or any assistance from others. If you have any questions, feel free to ask me. Please do your work on separate paper; you should staple this sheet to your work (put this on top) and turn in everything together.

(1) The following is a clip from the game show **The Joker's Wild**.

`https://www.youtube.com/watch?v=r0iQ2-Zciys`

Watch the clip (the first part only) and evaluate the contestant's choices. Would you have done the same?

Some things to note:

- There are three wheels, and he must spin all three of them at the same time. Each has twelve slots, with the following values: 25 (twice), 50 (twice), 75 (twice), 100 (twice), 150 (twice), 200 (once), the devil.
- You will *not* be able to compute the probability that the contestant gets to $1,000 (and therefore wins the big prize) in a short computation. You will need to introduce simplifying assumptions and/or guesstimates to give a reasonable solution to the problem. That said, your solution should certainly involve some actual calculations.

Solution. First of all, let's catalogue what happens. His first spin is $200, and he elects to go on. His second spin is another $125, so $325 total. (This is in 1974, so multiply the figures by about six to get today's dollars.) He elects to spin again, despite someone in the audience (perhaps his wife or girlfriend?) urging him to stop. He gets the devil and loses.

We compute the expected value of another spin, assuming no devils appear. Each wheel contributes

$$\frac{1}{11} \cdot \Big(2 \cdot 25 + 2 \cdot 50 + 2 \cdot 75 + 2 \cdot 100 + 2 \cdot 150 + 200\Big) = \frac{1000}{11}$$

and there are three of them, so the total positive contribution is $\frac{3000}{11}$, which is a little bit less than 300. Let's round it off to 275.

The probability of at least one devil is $1 - (\frac{11}{12})^3$. You can compute this exactly with a calculator, but it is approximately $\frac{1}{4} = \frac{1}{12} + \frac{1}{12} + \frac{1}{12}$. (This is only inaccurate because two or more of the wheels could show devils.)

So, with probability approximately $\frac{1}{4}$, he loses $325, and with probability approximately $\frac{3}{4}$, he gains (on average) $275. The expected value of another spin is thus approximately

$$\frac{3}{4} \times 275 + \frac{1}{4} \times (-325).$$

Even without a calculator it is apparent that this is much larger than zero. So, if the contestant takes one more spin, and doesn't mind taking a risk, it is clearly a good idea to spin again. (Moreover, we don't even have to factor in the additional bonus prize if he eventually goes up to $1000 to conclude this! It is wise to spin again, even ignoring the bonus. And, besides, who actually wants that neon crap anyway?)

(2) A contestant plays a game of **Rat Race** on The Price Is Right. She prices two of the three small items correctly, and gets to race two rats.

The rules for Rat Race are recalled at the end of the exam.

(a) What is the probability that she wins the car (the first place prize)? (**Solution:** $\frac{2}{5}$)

(b) What is the probability that she wins the second place prize? (**Solution:** $\frac{2}{5}$)

(c) What is the probability that she wins both of the top two prizes? (**Solution:** $\frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$.)

(d) What is the probability that she wins the second place prize, if she wins the car?

Solution. This is our conditional probability formula: the probability of winning both prizes, divided by the probability of winning the car. We have $\frac{1/10}{2/5} = \frac{1}{4}$.

Alternatively, we can reason more directly. Given one rat in first position, the other rat can be in second, third, fourth, or fifth position – and each of these is equally likely. So again we get $\frac{1}{4}$.

(e) If the three prizes are worth $14,000, $2,000, and $1,000 respectively, what is the expected value of the game?

Solution. She has a $\frac{2}{5}$ probability of winning each prize. So the expected value is

$$\frac{2}{5} \times 14000 + \frac{2}{5} \times 2000 + \frac{2}{5} \times 1000 = \frac{2}{5} \times 17000 = 6800.$$

Note that these probabilities are not independent, but we don't need them to be for this computation to work!

(3) You keep flipping a coin until you flip tails once or flip heads four times. You get a dollar for each time you flip heads.

(a) What is the probability that you flip exactly two heads?

Solution. This happens if you flip heads, heads, and tails in that order. So $(1/2)^3 = \frac{1}{8}$.

(b) What is the expected value of the game?

Solution. By similar reasoning, we flip no heads with probability $\frac{1}{2}$, one head with probability $\frac{1}{4}$, two heads with probability $\frac{1}{8}$, three heads with probability $\frac{1}{16}$, four heads with probability $\frac{1}{16}$.

So the expected value is

$$\frac{1}{2} \times 0 + \frac{1}{4} \times 1 + \frac{1}{8} \times 2 + \frac{1}{16} \times 3 + \frac{1}{16} \times 4 = \frac{15}{16}.$$

A little less than a dollar.

(4) In a different game, you now simply flip a coin four times. You get a dollar for each time you flip heads.

(a) What is the probability that you flip exactly two heads?

Solution. There are sixteen possible outcomes (HHHH, HHHT, HHTH, etc.) and six of these have exactly two heads. You can count them explicitly or observe that the answer is $C(4,2)$ and compute that. So $\frac{6}{16} = \frac{3}{8}$.

(b) What is the expected value of the game?

Solution. 50 cents for each flip, so two bucks.

(5) You play a dice game where you get to roll one die up to two times: you roll the die, and choose to either keep the result or reroll the die (once only) and keep that.

(a) Suppose first that you win $10,000 if you roll a six, and nothing otherwise. What are your odds of winning?

Solution. The probability of failing to roll a six twice in a row is $(5/6)^2 = \frac{25}{36}$, so $\frac{11}{36}$.

(b) Now suppose that you win $1,000 times the result of your die. With optimal strategy, what is the expected value of the game?

Solution. The average die roll is 3.5, so if you roll a 1, 2, 3 on the first die you will reroll and win $3,500 on average. So the expected value is

$$\frac{1}{6} \times 4000 + \frac{1}{6} \times 5000 + \frac{1}{6} \times 6000 + \frac{1}{2} \times 3500 = 4250.$$

(6) You are dealt a two card poker hand at random in a game of Texas Hold'em.

(a) Compute the probability that you are dealt a pair.

Solution. Number of hands: $C(52,2) = \frac{52 \cdot 51}{2} = 1326$.

Number of pairs: 13 ranks times $C(4,2) = 6$ pairs of each rank, so $13 \times 6 = 78$. Probability $\frac{78}{1326}$.

(b) Compute the probability that your two cards are of the same suit.

Solution. There are four suits, and for each suit there are $C(13,2) = \frac{13 \cdot 12}{2} = 78$ pairs of cards in that suit. So $\frac{4 \cdot 78}{1326}$.

(c) A *suited connector* is two consecutive cards of the same suit. (An ace is consecutive with both a two and a king.) Compute the probability that you are dealt a suited connector.

Solution. The top card can be any of the 52 cards in the deck, and then this determines the bottom card. So $\frac{52}{1326}$.

(d) Compute the combined probability that you are dealt any one of the above hands.

Solution. Suited connectors are counted in the second possibility, so add the first two: $\frac{5 \cdot 78}{1326}$.

Rules for Rat Race (The Price Is Right): The game is played for three prizes: a small prize, a medium prize, and a car.

There is a track with five wind-up rats (pink, yellow, blue, orange, and green). The contestant attempts to price three small items, and chooses one rat for each successful attempt. The rats then race. If she picked the third place rat, she wins the small prize; if she picked the second place rat, she wins the medium prize; if he picked the first place rat, she wins the car.

(Note that it is possible to win two or even all three prizes.)