Midterm Exam (with Solutions), SCHC 212, Frank Thorne (thornef@mailbox.sc.edu)

Wednesday, March 7, 2018

Please work without books, notes, calculators, or any assistance from others. If you have any questions, feel free to ask me. Please do your work on separate paper; you should staple this sheet to your work (put this on top) and turn in everything together.

(1) The following is a clip from the game show **Catch 21**.

`https://www.youtube.com/watch?v=c_kVCb1PVSw`

Watch the clip (the first part only) and evaluate the contestant's final decision to walk away. Would you have done the same? (Assume that you are single, so that "my wife would kill me" isn't a factor.)

Some things to note:

- *The rules as of the end of the game.* The contestant has won $5,000, and can either walk away or risk it for $25,000.

  He draws cards from a poker deck, one at a time, and to win $25,000 he must draw cards whose total value is seven. After each card: if his cards total seven, he wins $25,000 (total, including his $5,000); if his cards total more than seven, he 'busts' and loses his $5,000; as long as his cards total less than seven, he will be invited after each card to keep going or to walk away.

  He also holds one *power chip*. If he busts, he can use up his power chip, and draw another card in place of the card that busted him. (He must take this card, he cannot walk away immediately after using his power chip.)

- You should introduce simplifying assumptions, guesstimates, and/or approximations as needed. That said, your solution should certainly involve some actual calculations.

  Solution. There are only 42 cards left in the deck; the following 10 cards are missing: 2, 3, 4, 8, 9, 10, 10, J, J, Q. (If you simplified by imagining that all 52 cards are available, that doesn't change the solution too much.)

  What could happen on the first round?

  - **You immediately draw a 7.** There are 4 out of 42 available, so probability $\frac{4}{42} = 0.095$.
  - **You draw an eight or higher, use the power chip, and then draw a 7 on the second card.** There are 17 cards which are eight or higher, so probability $\frac{17}{42} \cdot \frac{4}{41} = 0.039$.
  - **You draw two cards in a row which are eight or higher, and bust.** Probability $\frac{17}{42} \cdot \frac{16}{41} = 0.158$.

– **Otherwise you draw a six or lower, including an ace** (either on the first card or after the power chip). This is the most likely outcome, with probability

$$1 - 0.095 - 0.039 - 0.158 = .708.$$

You then have the chance to walk away – let's assume for now that you take it.

Then the expected value of taking just one more round is

$$(0.095 + 0.039) \cdot 25000 + .708 \cdot 5000 = 6890.$$

This is substantially higher than 5000, so if you have the stomach for risk, it makes sense to take another round.

If the game reaches a second round, should you take it? Your odds of *winning* on the second round will be approximately the same as they were on the first – but the odds of busting go up, and the odds of getting another chance to walk away go down. So probably this doesn't make sense unless *maybe* you draw an ace or two.

But the above computation does show that you increase your expected value if you take one more turn.

(2) You play a game where you flip coins and win \$5 for every heads you flip. In the first round, you flip three coins. You can then either keep your money, or throw it away and flip a second time. If you decide to flip a second time, then you are allowed to flip only two coins.

(a) What is the probability that you win \$15?

Solution. You have to flip heads on all three coins the first flip, so $(1/2)^3 = 1/8$.

(b) What is the probability that you win nothing?

Solution. Assume that if you get any money at all you will take it and walk away. (This is *not* the only reasonable assumption, and so not necessarily the only correct answer!) You have to flip tails on all five coins, so $(1/2)^5 = 1/32$.

(c) What is the probability that you win at least \$5?

Solution. This is the opposite scenario from winning nothing, so $1 - 1/32 = 31/32$.

(d) Assuming that you play to maximize your expected value, describe an optimal strategy for playing this game, and compute the expected value of following this strategy.

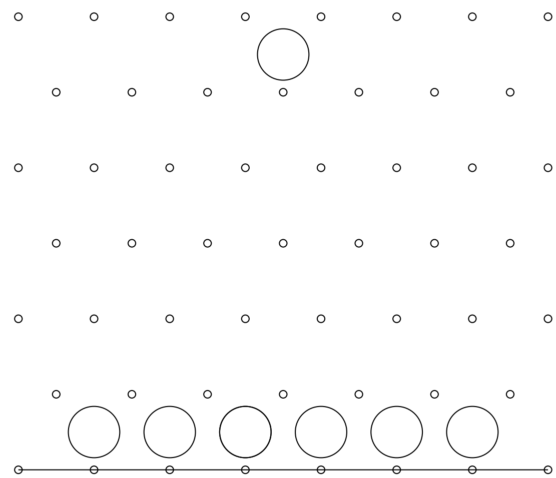
Solution. First note that if you throw away the first round, the expected value of going to the second round is $2 \times \frac{1}{2} \times \$5 = \$5$.

If you win \$15 or \$10 on the first round, then you can't do better so you take it. If you win \$5 on the first round, you either keep it, or throw it away and have an expected value of \$5 from the second round. (Either choice leads to the same expected value.) If you win nothing, you get an expected value of \$5 from the second round.

The total expected value is

$$\frac{1}{8} \cdot 15 + \frac{3}{8} \cdot 10 + \frac{3}{8} \cdot 5 + \frac{1}{8} \cdot 5 = \$8.125.$$

(3) Consider the following simplified game of Plinko:



A puck starts at the circled location on the top. In each of five stages, the puck goes left or right with probability $\frac{1}{2}$ each until it lands in one of the six circled spots on the bottom.

(a) How many different paths for the puck are possible?

Solution. There are 2 possibilities for each of five stages, so $2^5 = 32$.

(b) What is the probability that the puck lands in the far left circle?

Solution. The puck has to go left five times in a row, so $\frac{1}{32}$.

(c) What is the probability that the puck lands in the third circle from the left?

Solution. Write out the fifth row of Pascal's triangle: 1, 5, 10, 10, 5, 1. These, when you divide by 32 (the total number of paths) give the probabilities of landing in each circle. So $\frac{10}{32} = \frac{5}{16}$.

(d) Suppose that the six circles are worth payoffs of \$-5, \$-3, \$-1, \$1, \$3, and \$5 from left to right. (Negative numbers mean you lose money.) What is the payoff of playing the game? Explain.

Solution 1. By the probabilities above, the expected value is

$$\frac{1}{32} \cdot (-5) + \frac{5}{32} \cdot (-3) + \frac{10}{32} \cdot (-1) + \frac{10}{32} \cdot 1 + \frac{5}{32} \cdot 3 + \frac{1}{32} \cdot 5 = 0.$$

Solution 2. Because of the symmetry in Pascal's triangle, the probability of landing in a positive slot on the right equals the probability of landing in the corresponding negative slot on the left. Therefore their contributions to the expected value cancel out, and the total expected value is zero.

Solution 3. If you look at the initial position of the puck relative to the spots on the bottom, it is halfway between the -1 and 1 positions, so an average of zero.

Every time the puck goes left, you lose a dollar (because the puck goes left half a circle), and every time the puck goes right, you gain a dollar. The expected value of each stage is thus $\frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$. By linearity of expectation, the expected value of all five stages together, and hence the game as a whole, is zero.